

# Bethe Ansatz in Stringy Sigma Models

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## Abstract

We compute the exact S-matrix and give the Bethe ansatz solution for three sigma-models which arise as subsectors of string theory in  $\text{AdS}_5 \times S^5$ : Landau-Lifshitz model (non-relativistic sigma-model on  $S^2$ ), Alday-Arutyunov-Frolov model (fermionic sigma-model with  $\mathfrak{su}(1|1)$  symmetry), and Faddeev-Reshetikhin model (string sigma-model on  $S^3 \times R$ ).

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## 1 Introduction

According to the AdS/CFT correspondence [1] solving four-dimensional  $\mathcal{N} = 4$  super-Yang-Mills (SYM) theory amounts to quantization of type IIB superstrings on the  $\text{AdS}_5 \times S^5$  background. This potentially easier problem has so far resisted solution, partly because the sigma-model on  $\text{AdS}_5 \times S^5$  [2] is of the Green-Schwarz type and has well-known difficulties in the conformal gauge. The sigma model, however, is integrable [3] which gives us hope to solve it with the help of Bethe ansatz. There is a mounting evidence that the non-perturbative spectrum of AdS/CFT is indeed described by some sort of Bethe equations [4, 5]. The strongest evidence comes from studying the spectrum of anomalous dimensions in the SYM theory [6, 7, 8] and from analyzing classical solutions in the sigma-model [9, 10, 11], which describe spinning strings in  $\text{AdS}_5 \times S^5$  [12].

The Bethe ansatz [13, 14, 15] is a common method to solve integrable models, which encodes the whole spectrum of the system in a set of algebraic equations. The Bethe

equations can be interpreted in terms of factorized particle scattering, where the “Bethe particles” do not necessarily coincide with the physical degrees of freedom of the theory. In such an interpretation, the Bethe equations arise as quantization conditions for the momenta of the particles in a box of linear size  $L$ :

$$e^{ip_j L} = \prod_{k \neq j} e^{-i\Delta(p_j, p_k)} , \quad (1.1)$$

where  $\Delta(p_j, p_k)$  is the phase shift experienced by the  $j$ th particle when it scatters off the  $k$ th particle. The energy of a Bethe state is the sum of the single-particle energies:

$$E_{\{p_j\}} = \sum_j \varepsilon(p_j) . \quad (1.2)$$

The energy spectrum is thus completely determined by the two-body scattering and the one-body dispersion relation. There are no genuine multi-body interactions [16].

The SYM theory is an example where the Bethe particles are very different from the excitations in the four-dimensional space-time. Here, the Bethe equations, which determine the planar spectrum of anomalous dimension, were derived by interpreting single trace operators in SYM as quantum states of abstract spin chains [6]. The Bethe particles are waves that propagate along the spin-chain associated with a local operator in the SYM, but more recent results [17] suggest that the story might be different (and far more interesting!) at the non-perturbative level. On the string-theory side, the evidence in favor of the Bethe-ansatz structure of the spectrum comes from the two sources: (i) The classical solutions of the string sigma-model can be parameterized by the integral equations of Bethe type [9, 10] and (ii) the leading quantum corrections in the near-BMN limit of the  $\text{AdS}_5 \times S^5$  geometry [18] can be parameterized by a set Bethe equations [19, 20]. The quantum string Bethe equations, which were conjectured on the basis of these two observations [19, 20, 8, 21], receive corrections at higher orders in the sigma-model coupling [22] and thus contain infinitely many unknown parameters. Deriving Bethe equations for quantum string in  $\text{AdS}_5 \times S^5$  from first principles is an open problem. In particular it is not quite clear what degrees of freedom of the string are represented by the Bethe particles.

The string sigma-model on  $\text{AdS}_5 \times S^5$  is a rather complicated two-dimensional field theory. Similar but simpler systems, such as the  $\text{Osp}(2m+2|2m)$  coset sigma-model, were solved by Bethe ansatz [23], because many two-dimensional factorized S-matrices are known exactly [16]. For other models a direct relationship to spin chains can be established [24]. The idea that we would like to put forward in this paper, and test on a number of simplified models, is to derive the Bethe equations for quantum strings by explicitly computing the two-body scattering matrix on the world-sheet. We use relatively simple and rather standard methods of quantum field theory to do that. This will allow us to obtain the Bethe equations for several two-dimensional field theories that arise as reductions of the  $\text{AdS}_5 \times S^5$  sigma-model. We should mention that the S-matrix approach was quite successfully used on the gauge-theory side of the AdS/CFT correspondence [20, 25, 26].

We demand (rather than prove) quantum integrability and factorization of the S-matrix as necessary prerequisites. With these assumptions in mind, our method provides the means to *derive* the Bethe equations of an integrable theory at the quantum level. A more rigorous approach to Bethe ansatz is the quantum inverse scattering method [27] which fully exploits the rich algebraic structure associated with integrability.

The first model we consider is a non-relativistic sigma model on  $S^2$  (the Landau-Lifshitz model). This model describes fast moving strings on  $S^3 \times R$  and arises as a low-energy effective theory of the Heisenberg ferromagnet (the connection that plays an important role in the AdS/CFT correspondence [28]). We derive the Bethe equations for the LL model in Sec. 3 by direct computation of the S-matrix. In Sec. 4 we consider the fermionic sigma-model that arises from the  $\mathfrak{su}(1|1)$  reduction of the AdS string [29]. Finally, in Sec. 5 we consider again string theory on  $S^3 \times R$ , but this time without making the low energy approximation. This model was introduced by Faddeev and Reshetikhin [30] and is closely related to the  $\mathfrak{su}(2)$  principal chiral field, whose Bethe-ansatz solution was obtained in [31]. The lattice-regularized quantum version of the FR model was solved in [30]. We derive the Bethe equations directly in the continuum.

## 2 Preliminaries

The general idea is to derive the Bethe equations for quantum strings from scattering computations in the world-sheet theory. Let us sketch this idea for the example of a single scalar field on the world-sheet. In the usual Hamiltonian approach to the Bethe equations, one constructs eigenstates which have the structure of scattering plane-waves. In the two-particle sector such an eigenstate of the Hamiltonian looks like

$$|p p'\rangle = \int dx dx' \underbrace{[\theta(x' - x) + \theta(x - x')S(p, p')]}_{\chi(x, x')} e^{ipx + ip'x'} \varphi^\dagger(x) \varphi^\dagger(x') |0\rangle. \quad (2.1)$$

It is parameterized by two momenta  $p$  and  $p'$ . We will always label the momenta such that  $p > p'$ , so that the first term in the wave function is the incoming wave, and the second term is the scattered wave. Imposing the periodicity condition on the wave function:  $\chi(0, x') = \chi(L, x')$  and  $\chi(x, 0) = \chi(x, L)$ , one finds that the momenta of the particles are quantized according to

$$e^{iLp} = S(p', p) \quad , \quad e^{iLp'} = S(p, p') \quad , \quad (2.2)$$

where  $S(p', p) = 1/S(p, p')$ . This is a particular case of (1.1), for which integrability is actually not required. The distinguishing feature of integrable models is that the multi-body wave function is two-particle reducible [32, 15]. The periodicity condition for an arbitrary multi-particle eigenstate of the Hamiltonian is then expressed in terms of the two-body phase shifts as in (1.1).

Diagonalization of the Hamiltonian, however, is not the most efficient way to calculate the scattering phase shifts in field theory. It is much easier to compute  $S(p, p')$  as the matrix element of the infinite time evolution operator  $\hat{S}$  between the two-particle scattering states:

$$\langle k k' | \hat{S} | p p' \rangle = S(p, p') \delta_+(p, p', k, k'). \quad (2.3)$$

Here we have introduced the notation

$$\delta_{\pm}(p, p', k, k') = (2\pi)^2 (\delta(p - k)\delta(p' - k') \pm \delta(p - k')\delta(p' - k)) . \quad (2.4)$$

This factor represents the conservation of individual momenta during the scattering process. In two dimensions, energy and momentum conservation allows two particles only to exchange their momenta. The relative sign between the two terms is plus for bosons and minus for fermions.

We will compute the S-matrix using Feynman diagrams. Note that the usual Feynman rules calculate the matrix element  $\mathcal{M}(p, p', k, k')$  as defined in

$$\langle k k' | (\hat{S} - \mathbf{1}) | p p' \rangle = i \mathcal{M}(p, p', k, k') (2\pi)^2 \delta^{(2)}(p^\mu + p'^\mu - k^\mu - k'^\mu) . \quad (2.5)$$

The energy-momentum conserving delta-function is different from (2.4) by a Jacobian  $1/(\partial\varepsilon/\partial p - \partial\varepsilon/\partial p')$ , which has to be taken into account when extracting the phase shift from the diagrammatic calculations.

In what follows we make use of (2.3) and derive the Bethe equations for three different theories that are of potential relevance for quantum strings in  $\text{AdS}_5 \times S^5$ .

### 3 Landau-Lifshitz model

The LL model is defined by the action

$$\mathcal{S} = \int d^2x \left[ C_t(\vec{n}) - \frac{1}{4}(\partial_x \vec{n})^2 \right] , \quad (3.1)$$

where  $\vec{n}$  is a three-dimensional unit vector:

$$\vec{n}^2 = 1 . \quad (3.2)$$

The action contains the non-local Wess-Zumino term

$$C_q(\vec{n}) = -\frac{1}{2} \int_0^1 d\xi \varepsilon_{ijk} n_i \partial_\xi n_j \partial_q n_k , \quad (3.3)$$

and is of the first order in time derivatives. The equations of motion that follow from (3.1) are

$$\partial_t n_i = \varepsilon_{ijk} n_j \partial_x^2 n_k . \quad (3.4)$$

The LL equation is completely integrable [33]. The quantum inverse scattering method for the LL model was discussed in [34].

The LL model is the low-energy effective field theory of the Heisenberg ferromagnet with Hamiltonian

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^L (1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}) , \quad (3.5)$$

which in the AdS/CFT context arises as the one-loop mixing matrix of scalar composite operators  $\text{tr}(Z^{L-M} W^M + \text{permutations})$  [6]. The Heisenberg equations of motion for

(3.5) reduce to (3.4) in the continuum limit  $L \rightarrow \infty$  if the spin operators are formally replaced by unit c-number vectors:  $\vec{\sigma}_l(\tau) \rightarrow \vec{n}(\tau, \sigma)$  with  $\sigma = 2\pi l/L$ . Alternatively, the action in the coherent-state path integral of the Heisenberg model in the continuum limit becomes [28]

$$\mathcal{S} = \frac{L}{2\pi} \int d\tau \int_0^{2\pi} d\sigma \left[ C(\vec{n}) - \frac{\lambda}{8L^2} (\partial_\sigma \vec{n})^2 \right]. \quad (3.6)$$

This is the same as (3.1) after the following rescalings:  $\sigma = 2\pi x/L$ ,  $\tau = 8\pi^2 t/\lambda$ . The spacial coordinate  $x$  in (3.1) now has periodicity  $L$ .

The LL model can be also derived from classical string theory on  $S^3 \times R$  in the limit of fast-moving strings. The details of the derivation, together with the precise matching to the mixing matrix for long operators in the SYM, can be found in [28]. Here we would like to view the LL model as a (1+1)-dimensional quantum field theory. We derive the Bethe equations for the non-perturbative spectrum of the LL model with the help of relatively simple perturbative calculations, which are similar to the perturbative calculation of the S-matrix in the closely non-linear Schrödinger model [35]. Quantum-mechanical (Hamiltonian) perturbation theory for the LL model was developed in [36, 37]. Here, we will use Feynman diagrams which greatly facilitates the calculation of the S-matrix.

In order to develop perturbation theory we first re-write the WZ term in the local form. For that we will need some properties of the WZ action, which we review here following [38]. The WZ term in (3.1) can be written as

$$\text{WZ}[\vec{n}] := \int dt C_t(\vec{n}) = -\frac{1}{4} \int \varepsilon_{ijk} n_i dn_j \wedge dn_k, \quad (3.7)$$

where  $d\vec{n} = \partial_t \vec{n} dt + \partial_\theta \vec{n} d\theta$ . A short calculation shows that

$$\frac{1}{2} \varepsilon_{ijk} n_i dn_j \wedge dn_k = \frac{dn_1 \wedge dn_2}{n_3} = d \left( \frac{n_1 dn_2 - n_2 dn_1}{1 + n_3} \right). \quad (3.8)$$

This identity allows one to write the WZ action in a local form, which comes at the price of losing manifest  $\text{SO}(3)$  invariance:

$$\text{WZ}[\vec{n}] = \frac{1}{2} \int dt \frac{\dot{n}_1 n_2 - \dot{n}_2 n_1}{1 + n_3}. \quad (3.9)$$

The next step is to solve the constraint (3.2) by expressing  $n_3$  in term of  $n_1$  and  $n_2$ . It is also convenient to make a field redefinition which gets rid of the non-linearities in the kinetic term [36], and also to combine  $n_1, n_2$  into a single complex scalar, since the complex field has the canonical non-relativistic propagator. The significance of this fact will become clear later. So we define

$$\varphi = \frac{n_1 + in_2}{\sqrt{2 + 2n_3}} \quad , \quad n_3 = 1 - 2|\varphi|^2. \quad (3.10)$$

Performing this change of variables in (3.9), (3.1), we find:

$$\begin{aligned} \mathcal{S} = \int d^2x \left[ \frac{i}{2} (\varphi^* \partial_t \varphi - \partial_t \varphi^* \varphi) - |\partial_x \varphi|^2 - \frac{1}{4} \frac{2 - |\varphi|^2}{1 - |\varphi|^2} [(\varphi^* \partial_x \varphi)^2 + (\partial_x \varphi^* \varphi)^2] \right. \\ \left. - \frac{1}{2} \frac{|\varphi|^4 |\partial_x \varphi|^2}{1 - |\varphi|^2} \right]. \end{aligned} \quad (3.11)$$

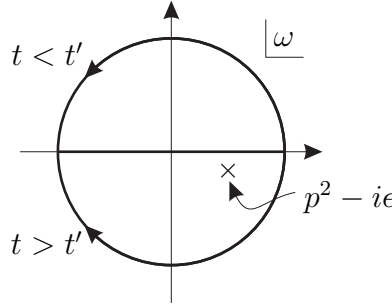


Figure 1: **Pole prescription in Landau-Lifshitz model.**

This action describes an interacting field theory of a single scalar field. Its non-relativistic character leads to some important non-renormalization properties.

First of all, the ground state is annihilated by the field operator:

$$\varphi(t, x)|0\rangle = 0 . \quad (3.12)$$

Since the equations of motion are of the first order in time derivatives, the field operator in the interaction picture is expanded in negative-frequency modes only:

$$\varphi(t, x) = \int \frac{dp}{2\pi} a_p e^{-ip^2 t + ipx} , \quad \varphi^*(t, x) = \int \frac{dp}{2\pi} a_p^\dagger e^{ip^2 t - ipx} , \quad (3.13)$$

where  $a_p, a_p^\dagger$  create and annihilate a particle with momentum  $p$  and energy<sup>1</sup>

$$\varepsilon(p) = \frac{\lambda}{8\pi^2} p^2 . \quad (3.14)$$

The operators  $a_p$  and  $a_p^\dagger$  obey canonical commutation relation normalized as

$$[a_p, a_{p'}^\dagger] = 2\pi \delta(p - p') . \quad (3.15)$$

Since the ground state is annihilated by  $\varphi(t, x)$ , the particles do not travel backwards in time and the propagator, accordingly, has only one pole in the momentum representation:

$$\begin{aligned} D(t, x) &= \langle 0 | T \varphi(t, x) \varphi^*(0, 0) | 0 \rangle = \begin{array}{c} \bullet \longleftarrow \bullet \\ (t, x) \qquad (0, 0) \end{array} \\ &= \int \frac{d\omega dp}{(2\pi)^2} \frac{i}{\omega - p^2 + i\epsilon} e^{-i\omega t + ipx} . \end{aligned} \quad (3.16)$$

The pole prescription, see Fig. 5, or again the fact that  $\varphi(t, x)$  annihilates the vacuum makes the coordinate-space propagator purely retarded:

$$D(t, x) = \theta(t) \sqrt{\frac{\pi}{it}} e^{\frac{ix^2}{4t}} . \quad (3.17)$$

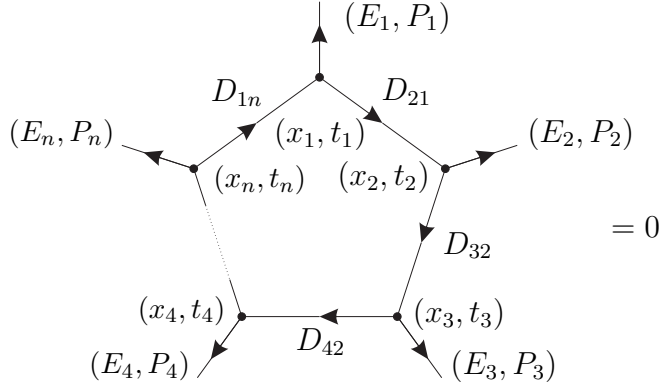


Figure 2: **Non-renormalization theorem.** Any closed loop of likewise oriented propagators  $D_{mn} \equiv D(t_m - t_n, x_m - x_n)$  vanishes.

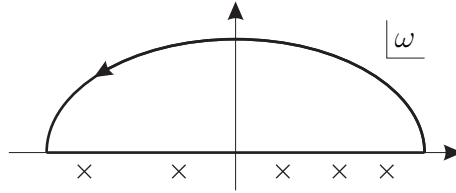


Figure 3: **Poles for a closed loop.** All propagators in a closed have their poles in the lower half-plane. Hence the integral (3.19) over the energy flowing around the loop vanishes.

We can now prove the following “non-renormalization theorem”: any diagram that contains a closed loop with arrows in the same direction vanishes, cf. Fig. 2. In the coordinate representation this follows from the fact that at least one propagator in the loop has a negative time argument. Consequently,

$$D(t_2 - t_1, x_2 - x_1)D(t_3 - t_2, x_3 - x_2) \cdots D(t_1 - t_n, x_1 - x_n) = 0. \quad (3.18)$$

In the momentum space representation, the integrand has poles only in the lower half-plane of complex  $\omega$ . The integral is then zero by the contour argument, cf. Fig. 3:

$$\int \frac{d\omega}{2\pi} \frac{i}{\omega - p^2 + i\epsilon} \frac{i}{\omega - E_1 - (p - P_1)^2 + i\epsilon} \cdots \frac{i}{\omega + E_n - (p + P_n)^2 + i\epsilon} = 0. \quad (3.19)$$

This theorem has three important consequences:

- The ground state energy is not renormalized:  $E_{\text{vac}} = 0$ . This is consistent with the fact that the ferromagnetic vacuum is the exact zero-energy eigenstate of the Heisenberg Hamiltonian (3.5) (cf. the discussion in [36]).

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<sup>1</sup>Recall that we have rescaled the time variable by  $8\pi^2/\lambda$ . Hence, the energy is the frequency multiplied by  $\lambda/8\pi^2$



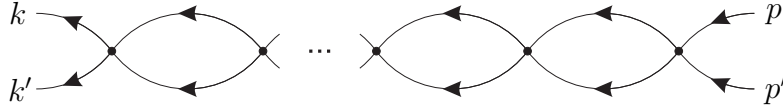


Figure 4: **Generic loop diagram for the two-body S-matrix.**

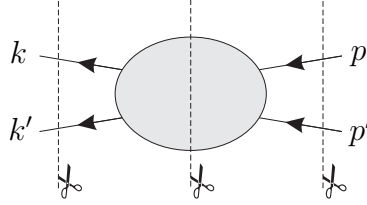


Figure 5: **Cutting a generic diagram.** Due to charge conservation the number of future directed propagators minus the number of past directed propagators has to be the same at any moment in time. Since any past directed propagator is identically equal to zero, the number of propagators at any cut is the same and is equal to the number of external incoming/outgoing legs.

- The one particle Green's function is not renormalized. Hence, the dispersion relation (3.14) does not receive quantum corrections.
- The two-body S-matrix is given by the sum of bubble diagrams (Fig. 4).

These properties are almost obvious. A formal proof can be given by cutting a generic diagram and counting intermediate propagators, as illustrated in Fig. 5.

The fact that the two-body S-matrix is determined by the sum of bubble diagrams in Fig. 4 has far reaching consequences. Since these diagrams contain only quartic vertices we may truncate the non-polynomial action (3.11) at the fourth order in the fields<sup>2</sup>:

$$\mathcal{L} = \frac{i}{2} (\varphi^* \partial_t \varphi - \partial_t \varphi^* \varphi) - |\partial_x \varphi|^2 - \frac{g}{2} [(\varphi^* \partial_x \varphi)^2 + (\partial_x \varphi^* \varphi)^2] + \mathcal{O}(\varphi^6) . \quad (3.20)$$

This is a very important simplification, which makes the all-loop computation feasible and allows to forget about other non-linear terms in the action. In (3.20) we have introduced a formal expansion parameter  $g$  to make the power-counting of perturbative series more transparent. We will set  $g = 1$  at the end of the calculation. In fact, observable quantities cannot depend on  $g$ , since this parameter can be eliminated by rescaling  $t$ ,  $x$  and  $\varphi$ .

Let us now derive the Bethe equations for the LL model (3.1) respectively (3.11), i.e.

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<sup>2</sup>The resulting Lagrangian is very similar to the one of the non-linear Schrödinger model given by  $\mathcal{L} = \frac{i}{2} (\varphi^* \partial_t \varphi - \partial_t \varphi^* \varphi) - |\partial_x \varphi|^2 - g(\varphi^* \varphi)^2$ . For comparison we recall the S-matrix of this model:  $S(p, p') = \frac{p-p'-ig}{p-p'+ig}$ , which can be computed by the same technique as we use here [35].

compute the two-particle S-matrix. The two relevant vertices, written in (3.20), are

$$-\frac{g}{2}(\varphi^* \partial_x \varphi)^2 \longrightarrow \begin{array}{c} k \nearrow \quad \nwarrow p \\ \times \\ k' \nwarrow \quad \nearrow p' \end{array} = 2ig p_1 p'_1, \quad (3.21)$$

$$-\frac{g}{2}(\partial_x \varphi^* \varphi)^2 \longrightarrow \begin{array}{c} k \nwarrow \quad \nearrow p \\ \times \\ k' \nearrow \quad \nwarrow p' \end{array} = 2ig k_1 k'_1. \quad (3.22)$$

The two-particle in- and out-states are defined as

$$|p p'\rangle = a^\dagger(p) a^\dagger(p') |0\rangle, \quad \langle k k'| = \langle 0| a(k') a(k). \quad (3.23)$$

The non-scattering part is given by

$$\langle k k'|p p'\rangle = \delta_+(p, p', k, k'), \quad (3.24)$$

where  $\delta_+$  was defined in (2.4). At tree-level we need to evaluate the following expression

$$\langle k k'|\hat{S}|p p'\rangle \Big|_g = \langle k k'| \left( -\frac{ig}{2} \right) \int dt dx [(\varphi^* \partial_x \varphi)^2 + (\partial_x \varphi^* \varphi)^2] |p p'\rangle. \quad (3.25)$$

There are four identical Wick contractions between the states and each vertex leading to the factor  $4(-kk' - pp')$ . The integration over  $t$  and  $x$  imposes energy and momentum conservation, which can be written as

$$(2\pi)^2 \delta(p^2 + p'^2 - k^2 - k'^2) \delta(p + p' - k - k') = \frac{1}{2(p - p')} \delta_+(p, p', k, k'). \quad (3.26)$$

Multiplying all factors together we find

$$\langle k k'|\hat{S}|p p'\rangle \Big|_g = 2ig \frac{pp'}{p - p'} \delta_+(p, p', k, k'). \quad (3.27)$$

Including the non-scattering part (3.24), we can read off the S-matrix from comparison with (2.3) and find to first order

$$S(p, p') = 1 + 2ig \frac{pp'}{p - p'} + \mathcal{O}(g^2). \quad (3.28)$$

In App. A we compute all higher order corrections and find for arbitrary  $n$ , cf. (A.13):

$$\langle k k'|\hat{S}|p p'\rangle \Big|_{g^n} = 2 (ig)^n \left( \frac{pp'}{p - p'} \right)^n \delta_+(p, p', k, k'). \quad (3.29)$$

Summing this result for all  $n$  and adding the non-scattering part yields the exact S-matrix for the LL model

$$S(p, p') = 1 + 2 \sum_{n=1}^{\infty} (ig)^n \left( \frac{pp'}{p - p'} \right)^n = \frac{\frac{1}{p} - \frac{1}{p'} - ig}{\frac{1}{p} - \frac{1}{p'} + ig}, \quad (3.30)$$

where the expansion parameter  $g$  can now be set equal to unity.

The Bethe equations resulting from this S-matrix are most conveniently written in terms of the spectral variable  $u = 1/p$ :

$$e^{\frac{iL}{u_j}} = \prod_{k \neq j} \frac{u_j - u_k + i}{u_j - u_k - i}. \quad (3.31)$$

The energy and the momentum are then given by

$$E = \sum_j \frac{\lambda}{8\pi^2 u_j^2}, \quad P = \sum_j \frac{1}{u_j}. \quad (3.32)$$

The Bethe equations for the spectrum of the Heisenberg Hamiltonian (3.5) are very similar [14]. They are obtained by replacing  $1/u \rightarrow \pi - 2 \arctan u$  and  $1/u^2 \rightarrow 1/(u^2 + 1/4)$  in the Bethe equations and the dispersion relation above. The difference disappears at large  $u$  (small momenta), as expected. In particular, the spectrum of the low-energy modes with  $p \sim 1/u \sim 1/L$ , is the same for both models and can be calculated from the approximate equation

$$\frac{L}{u_j} - 2\pi n_j = \sum_{k \neq j} \frac{2}{u_j - u_k}, \quad (3.33)$$

up to and including finite-size  $\mathcal{O}(1/L)$  corrections [6]. As shown in [37], the spectra for the two models start to deviate from one another at order  $\mathcal{O}(1/L^2)$ .

The classical limit is achieved by assigning a macroscopic number of Bethe roots to a finite set of mode numbers  $n_I$ . Then (3.33) becomes an integral equation [39]:

$$\oint \frac{dy \rho(y)}{x - y} = \frac{1}{x} - 2\pi n_I, \quad (3.34)$$

where

$$\rho(x) = \frac{1}{L} \sum_j \delta\left(x - \frac{u_j}{L}\right). \quad (3.35)$$

It can be shown that (3.34) describes all time-periodic classical solutions of the LL equation (3.4) [9]. This classical limit of the Bethe equations is obviously the same for the LL and the Heisenberg models.

The spectrum that follows from Bethe equations (3.31) indicates a rather unexpected instability. The bound states of elementary excitations are described in integrable field theories by string solutions of the Bethe equations. For instance, the 2-string configuration is

$$u_{1,2} = v \pm \frac{i}{2}. \quad (3.36)$$

It becomes an exact solution of the Bethe equations in the strict thermodynamic ( $L \rightarrow \infty$ ) limit. The energy and the momentum of the 2-string are:

$$\begin{aligned} E_{2\text{-string}} &= \frac{\lambda}{8\pi^2} \left[ \frac{1}{\left(v + \frac{i}{2}\right)^2} + \frac{1}{\left(v - \frac{i}{2}\right)^2} \right] = \frac{\lambda}{4\pi^2} \frac{v^2 - \frac{1}{4}}{\left(v^2 + \frac{1}{4}\right)^2}, \\ P_{2\text{-string}} &= \frac{1}{v + \frac{i}{2}} + \frac{1}{v - \frac{i}{2}} = \frac{2v}{v^2 + \frac{1}{4}}. \end{aligned} \quad (3.37)$$

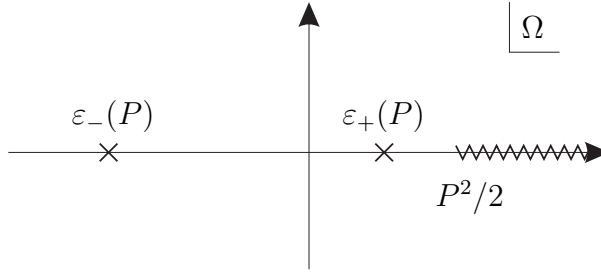


Figure 6: **Analytic structure of the two-particle Green's function (3.40).**

The momentum of the 2-string is always smaller than 2, so the 2-string solutions of the Bethe equations describe two kinds of excitations with the dispersion relations

$$\varepsilon_{\pm}(p) = \frac{\lambda}{8\pi^2} \left( \pm 2\sqrt{4 - p^2} - 4 + p^2 \right). \quad (3.38)$$

The two branches arise from  $v \gtrless 1/2$ . The  $\varepsilon_-$  branch has negative energy.

The existence of negative-energy states seems to signal a vacuum instability. Such an instability is absent in the Heisenberg spin chain, and its appearance in the LL model is rather puzzling. We have no simple explanation for this phenomenon, but we can demonstrate that the negative-energy states do arise as poles of the Green's functions. Consider, for instance, the Green's function of a composite operator

$$\mathcal{O} = (\varphi^* \partial_x \varphi)^2 + (\partial_x \varphi^* \varphi)^2, \quad (3.39)$$

which is computed in App. A.3:

$$\langle 0 | T \mathcal{O}(t, x) \mathcal{O}(0, 0) | 0 \rangle = \frac{i}{4} \int \frac{d\Omega dP}{(2\pi)^2} e^{-i\Omega t + iPx} \frac{(P^2 - \Omega)^2}{P^2 - \Omega - 2\sqrt{P^2 - 2\Omega}}. \quad (3.40)$$

As a function of  $\Omega$ , it has a two-particle cut from the threshold at  $\Omega = P^2/2$  to infinity and two poles at  $\frac{\lambda}{8\pi^2}\Omega = \varepsilon_{\pm}(P)$  (Fig. 6).

## 4 Alday-Arutyunov-Frolov model

In this section we derive the Bethe equations for an integrable system of two-dimensional fermions which was recently found by Alday, Arutyunov and Frolov [29]. This model is a new integrable quantum field theory whose Bethe-ansatz solution is not known. Classical integrability of the model was demonstrated in [29] by construction of a Lax representation for the classical equations of motion. The model arises as a consistent truncation of the classical  $\text{AdS}_5 \times S^5$  superstring to the  $\mathfrak{su}(1|1)$  subsector in the temporal gauge<sup>3</sup>.

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<sup>3</sup>In the light-cone gauge the  $\mathfrak{su}(1|1)$  sector is described by free world-sheet fermions [40].

The AAF model is a theory of an interacting Dirac (two-component complex) fermion in two dimensions<sup>4</sup>:

$$\mathcal{S} = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi L/\sqrt{\lambda}} d\sigma \int d\tau \left[ -\frac{i}{2}(\bar{\psi}\gamma^\alpha\partial_\alpha\psi - \partial_\alpha\bar{\psi}\gamma^\alpha\psi) + \bar{\psi}\psi \right. \\ \left. - \frac{1}{4}\varepsilon^{\alpha\beta}(\bar{\psi}\partial_\alpha\psi\bar{\psi}\gamma^3\partial_\beta\psi - \partial_\alpha\bar{\psi}\psi\partial_\beta\bar{\psi}\gamma^3\psi) \right. \\ \left. + \frac{1}{8}\varepsilon^{\alpha\beta}(\bar{\psi}\psi)^2\partial_\alpha\bar{\psi}\partial_\beta\psi \right]. \quad (4.1)$$

The action is manifestly Lorentz-invariant. The Lorentz metric is taken in the  $(+1, -1)$  signature and the notation  $\vec{x}$  refers to both components  $(x^0, x^1) \equiv (\tau, \sigma)$ . Below, we will use the scalar product  $\vec{a} \cdot \vec{b} = g^{\alpha\beta}a_\alpha b_\beta$  and the “vector” product  $\vec{a} \times \vec{b} = \varepsilon^{\alpha\beta}a_\alpha b_\beta$ , where the epsilon tensor satisfies  $\varepsilon^{01} = \varepsilon_{10} = +1$ . The explicit form of the Dirac matrices is given in (B.2) and the conjugate spinor is defined as  $\bar{\psi} = \psi^\dagger\gamma^0$ .

In the action (4.1) all quantities  $(\psi, \sigma, \tau, L, \lambda)$  are dimensionless. However, we prefer to write it in a way which allows us to assign canonical mass dimensions to the field and the coupling constants. Therefore we rescale the world-sheet coordinates by  $x^\alpha \rightarrow \frac{\sqrt{\lambda}}{2\pi}x^\alpha$  and introduce the parameters

$$m = \frac{2\pi}{\sqrt{\lambda}} \quad , \quad g = \frac{\pi}{2\sqrt{\lambda}}. \quad (4.2)$$

Then the space integral runs from 0 to  $L$  and the Lagrangian can be written as<sup>5</sup>

$$\mathcal{L} = -\bar{\psi}(i\partial - m)\psi \\ - \frac{g}{m^2}\varepsilon^{\alpha\beta}(\bar{\psi}\partial_\alpha\psi\bar{\psi}\gamma^3\partial_\beta\psi - \partial_\alpha\bar{\psi}\psi\partial_\beta\bar{\psi}\gamma^3\psi) \\ + \frac{4g^2}{m^3}\varepsilon^{\alpha\beta}(\bar{\psi}\psi)^2\partial_\alpha\bar{\psi}\partial_\beta\psi. \quad (4.3)$$

For the purpose of power counting, we may assign the following mass dimensions:  $[\psi] = \frac{1}{2}$ ,  $[L] = [\vec{x}] = -1$ ,  $[m] = 1$ ,  $[g] = 0$ . Clearly,  $m$  plays the role of a mass parameter and  $g$  the role of a coupling constant. The model is integrable for any  $m$  and  $g$ . It is not renormalizable by power-counting, but as conjectured in [29], the symmetries (integrability being one of them) can take care of renormalization and render the model renormalizable in perturbation theory, the same way symmetries make two-dimensional non-linear sigma-models renormalizable. In this sense, the AAF model is a fermionic counterpart of the non-linear *sigma*-models.

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<sup>4</sup>We use the Lagrangian (5.7) from [29], take the scaling (5.6) into account and denote the angular momentum of the string by  $L$  instead of  $J$ . In addition we shifted  $\sigma \rightarrow \sigma + 2\pi L/\sqrt{\lambda}$  to get positive limits for the integration. Finally we drop the constant term in the action, which amounts to shifting the origin of the energy scale.

<sup>5</sup>Now this theory resembles the massive Thirring model:  $\mathcal{L} = \bar{\psi}(i\partial - m)\psi - \frac{g}{2}\bar{\psi}\gamma^\alpha\psi\bar{\psi}\gamma_\alpha\psi$ . For later comparison we recall the S-matrix of this theory:  $S(\theta, \theta') = \frac{1 - i\frac{g}{2}\tanh\frac{\theta - \theta'}{2}}{1 + i\frac{g}{2}\tanh\frac{\theta - \theta'}{2}}$ . We checked that this S-matrix can be reproduced by the same diagrammatic calculation as we do below for the AAF model.

One cannot compute the physical S-matrix of the AAF model directly by resumming diagrams, as we did for the LL model, because the usual relativistic propagator is not purely retarded and loop corrections do not cancel. This difficulty is not fatal and can be overcome by the trick that dates back to [41] and was used by Bergknoff and Thacker to solve the massive Thirring model [42]. The idea is to use pseudo-vacuum (instead of the true ground state) as a reference state for quantization. The pseudo-vacuum, by definition, is a state annihilated by the field operator:

$$\psi(x)|0\rangle = 0. \quad (4.4)$$

All anti-particle levels in the pseudo-vacuum are left empty. Quantizing in the pseudo-vacuum can thus be interpreted as applying an infinite negative chemical potential to the system. The scattering S-matrix of excitations over the pseudo-vacuum can be computed exactly by essentially the same method as in any non-relativistic theory. This “bare” S-matrix can be then used to write down the Bethe equations. By solving the Bethe equations one can fill back the Dirac sea and reconstruct the true ground state. This is a pretty standard way to solve relativistic integrable field theories [15]. In principle it allows one to derive the spectrum of physical states and their S-matrix [43] from first principles. The price to pay is that the solution of the Bethe equations that describes the ground state and the solutions that describe physical excitations are already rather complicated. It is obvious that the filling of the Dirac sea drastically changes the spectrum of excitations and their S-matrix.

By solving the free equations of motion we find the following mode expansion of the fields

$$\psi(\vec{x}) = \int \frac{dp_1}{2\pi} [a(p_1)u(p_1)e^{-i\vec{p}\cdot\vec{x}} + b(-p_1)v(-p_1)e^{i\vec{p}\cdot\vec{x}}], \quad (4.5)$$

$$\bar{\psi}(\vec{x}) = \int \frac{dp_1}{2\pi} [a^\dagger(p_1)\bar{u}(p_1)e^{i\vec{p}\cdot\vec{x}} + b^\dagger(-p_1)\bar{v}(-p_1)e^{-i\vec{p}\cdot\vec{x}}], \quad (4.6)$$

where  $p_0 = \sqrt{p_1^2 + m^2}$ . The spinors  $u(p_1)$  and  $v(p_1)$  are defined in App. B.1. The oscillators obey the standard commutation relation:

$$\{a(p_1), a^\dagger(p'_1)\} = 2\pi\delta(p_1 - p'_1) \quad , \quad \{b(p_1), b^\dagger(p'_1)\} = 2\pi\delta(p_1 - p'_1). \quad (4.7)$$

The operators  $a^\dagger(p_1)$  and  $b^\dagger(p_1)$  create excitations with momentum  $p_1$  and with energies  $+\sqrt{p_1^2 + m^2}$  and  $-\sqrt{p_1^2 + m^2}$ . We define the pseudo-vacuum  $|0\rangle$  of the theory as the state satisfying

$$a(p_1)|0\rangle = b(p_1)|0\rangle = 0 \quad \text{for all } p_1. \quad (4.8)$$

We call the excitations created by  $a^\dagger$  and  $b^\dagger$  on top of this state pseudo-particles with positive and negative energy, respectively. The physical vacuum  $|\Omega\rangle$  is obtained from  $|0\rangle$  by exciting all negative energy modes, i.e. by filling all holes in the Dirac sea.

The major advantage of quantizing the theory in the pseudo-vacuum is the absence of anti-particles, which implies the same non-renormalization theorems as in the LL model of Sec. 3. Since the field operator  $\psi(\vec{x})$  annihilates the pseudo-vacuum, the propagator

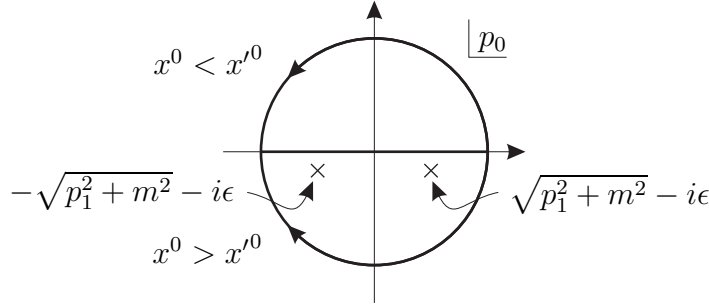


Figure 7: **Pole prescription in Alday-Arutyunov-Frolov model.**

is purely retarded:

$$\begin{aligned}
D(\vec{x} - \vec{x}') &:= \langle 0 | T \psi(\vec{x}) \bar{\psi}(\vec{x}') | 0 \rangle \\
&= (i\not{\partial} + m) \int \frac{d^2 p}{(2\pi)^2} \frac{i}{\vec{p}^2 - m^2 + i\epsilon(p_0)} e^{-i\vec{p} \cdot (\vec{x} - \vec{x}')} ,
\end{aligned} \tag{4.9}$$

where  $\epsilon(p_0) = \text{sign}(p_0)\epsilon$ . The fact that  $D(\vec{x} - \vec{x}')$  vanishes for  $x^0 < x'^0$  can easily be seen from the pole prescription which is depicted in Fig. 7. We mention that this pole prescription is relativistic invariant because  $\text{sign}(p_0)$  is invariant under orthochronous Lorentz transformations. Now, as both poles are in the lower half plane, we conclude by precisely the same reasoning as in the previous section that the closed loop of likewise oriented propagators vanishes, cf. Fig. 2. The most important consequence of this result is the fact that the two-body S-matrix is entirely determined by the four-valent vertices alone. These are given by

$$-\frac{g}{m^2} \varepsilon^{\alpha\beta} \bar{\psi} \partial_\alpha \psi \bar{\psi} \gamma^3 \partial_\beta \psi \longrightarrow \begin{array}{c} k \swarrow \quad \searrow p \\ \bullet \\ k' \swarrow \quad \searrow p' \end{array} = -\frac{ig}{m^2} \vec{p}' \times \vec{p} \, \mathbf{1} \otimes \gamma^3 , \tag{4.10}$$

$$+\frac{g}{m^2} \varepsilon^{\alpha\beta} \partial_\alpha \bar{\psi} \psi \partial_\beta \bar{\psi} \gamma^3 \psi \longrightarrow \begin{array}{c} k \swarrow \quad \searrow p \\ \circ \\ k' \swarrow \quad \searrow p' \end{array} = +\frac{ig}{m^2} \vec{k}' \times \vec{k} \, \mathbf{1} \otimes \gamma^3 . \tag{4.11}$$

The unit matrix  $\mathbf{1}$  (represented by the black dot) connects the  $p$  and  $k$  legs, whereas  $\gamma^3$  (represented by the white dot) connects the  $p'$  and  $k'$  legs. The six-valent vertex is crucial for the factorization of the S-matrix and hence for the integrability of the theory, but it is of no relevance for the scattering of two pseudo-particles!

In the following we compute the S-matrix between two particles of type  $a^\dagger$  (which have positive energy). The scattering of particle of type  $b^\dagger$  can be obtained by analytic continuation to complex rapidities  $\theta$ , defined by

$$p_0 = m \cosh \theta \quad , \quad p_1 = m \sinh \theta . \tag{4.12}$$

Real rapidities  $\theta = \alpha \in R$  parameterize the mass-shell of the positive energy particles  $a^\dagger$ , and complex rapidities  $\theta = i\pi - \alpha$  with  $\alpha \in R$  parameterize the mass-shell of the

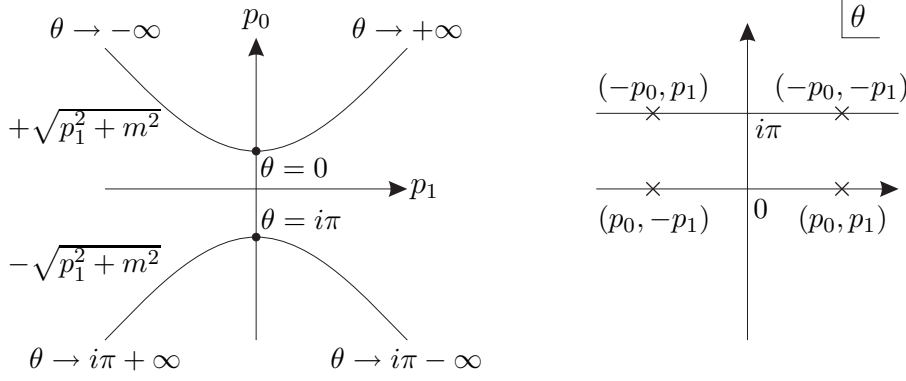


Figure 8: **Rapidity plane.** It is convenient to work with a complex rapidity  $\theta$  as particles ( $\text{Im } \theta = 0$ ) and anti-particles ( $\text{Im } \theta = \pi$ ) can be treated at the same time.

negative energy particles  $b^\dagger$ , cf. Fig. 8. As a meromorphic function of the rapidities, the S-matrix describes the scattering of both types of particle at the same time. Hence it is sufficient to consider the following in- and out-states:

$$|p_1 p'_1\rangle = a^\dagger(p_1) a^\dagger(p'_1) |0\rangle \quad , \quad \langle k_1 k'_1| = \langle 0| a(k'_1) a(k_1) . \quad (4.13)$$

The non-scattering part is given by

$$\langle k_1 k'_1 | p_1 p'_1 \rangle = \delta_-(p_1, p'_1, k_1, k'_1) , \quad (4.14)$$

where  $\delta_-$  was defined in (2.4). At tree-level we need to evaluate the following expression

$$\langle k_1 k'_1 | \hat{S} | p_1 p'_1 \rangle \Big|_g = \langle k_1 k'_1 | \left( -\frac{ig}{m^2} \right) \int d^2x \varepsilon^{\alpha\beta} [\bar{\psi} \partial_\alpha \psi \bar{\psi} \gamma^3 \partial_\beta \psi - \partial_\alpha \bar{\psi} \psi \partial_\beta \bar{\psi} \gamma^3 \psi] | p_1 p'_1 \rangle . \quad (4.15)$$

As the vertices do not have any symmetry, connecting the external lines with the vertex leads to  $2^3 = 8$  different terms. One of them is given by

$$- \left( -\frac{ig}{m^2} \right) (\vec{p}' \times \vec{p}) (\bar{u}(k'_1) \otimes \bar{u}(k_1)) (\mathbf{1} \otimes \gamma^3) (u(p_1) \otimes u(p'_1)) . \quad (4.16)$$

The others are obtained from this by exchange of the in-going particles ( $p \leftrightarrow p'$ ), by exchange of the out-going particles ( $k \leftrightarrow k'$ ) and by choosing the second vertex ( $\vec{p}' \times \vec{p} \rightarrow \vec{k}' \times \vec{k}$ ). All changes are accompanied by a change of the overall sign. The spacetime integral imposes energy and momentum conservation, which can be written as

$$(2\pi)^2 \delta(\vec{p} + \vec{p}' - \vec{k} - \vec{k}') = \frac{p'_0 p_0}{\vec{p}' \times \vec{p}} \delta_+(p_1, p'_1, k_1, k'_1) . \quad (4.17)$$

The  $\delta$ -functions in  $\delta_+(p_1, p'_1, k_1, k'_1)$ , cf. (2.4), can be used to convert the  $k$ 's into  $p$ 's. After this, the combination with the opposite relative sign  $\delta_-(p_1, p'_1, k_1, k'_1)$  forms. Eventually,



the whole expression boils down to

$$\begin{aligned} \langle k_1 k'_1 | \hat{S} | p_1 p'_1 \rangle \Big|_g &= \frac{2ig}{m^2} (\vec{p}' \times \vec{p}) (\bar{u}(p'_1) \otimes \bar{u}(p_1)) (\mathbf{1} \otimes \gamma^3 - \gamma^3 \otimes \mathbf{1}) (u(p_1) \otimes u(p'_1)) \\ &\quad \cdot \frac{p'_0 p_0}{\vec{p}' \times \vec{p}} \delta_-(p_1, p'_1, k_1, k'_1) . \end{aligned} \quad (4.18)$$

The scalar product of the spinors is given by (for useful identities see App. B.1)

$$\begin{aligned} (\bar{u}(p'_1) \otimes \bar{u}(p_1)) (\mathbf{1} \otimes \gamma^3 - \gamma^3 \otimes \mathbf{1}) (u(p'_1) \otimes u(p_1)) &= \frac{1}{2p'_0 p_0} \text{tr}(\not{p}' + m)(\not{p} + m)\gamma^3 \\ &= \frac{\vec{p}' \times \vec{p}}{p'_0 p_0} . \end{aligned} \quad (4.19)$$

Hence, at three level we find

$$\langle k_1 k'_1 | \hat{S} | p_1 p'_1 \rangle \Big|_g = \frac{2ig}{m^2} (\vec{p}' \times \vec{p}) \delta_-(p_1, p'_1, k_1, k'_1) . \quad (4.20)$$

Including the non-scattering part (4.14), we can read off the S-matrix as

$$S(p, p') = 1 + \frac{2ig}{m^2} (\vec{p}' \times \vec{p}) + \mathcal{O}(g^2) , \quad (4.21)$$

or in terms of rapidities (4.12) as

$$S(\theta, \theta') = 1 + 2ig \sinh(\theta - \theta') + \mathcal{O}(g^2) . \quad (4.22)$$

It is in fact possible to compute the S-matrix to all orders in  $g$ . For this computation to be possible, it is essential to quantize the theory in the pseudo-vacuum (4.8). This is because only in the pseudo-vacuum the higher loop diagrams are given by the sum of bubble diagrams, cf. Fig. 11. The full computation is nevertheless rather involved and therefore has been relegated to App. B.3. The higher order corrections are given in (B.21) and lead together with the non-scattering part to the exact S-matrix of the AAF model:

$$S(p, p') = \frac{1 + \frac{ig}{m^2} (\vec{p}' \times \vec{p})}{1 - \frac{ig}{m^2} (\vec{p}' \times \vec{p})} \quad \text{or} \quad S(\theta, \theta') = \frac{1 + ig \sinh(\theta - \theta')}{1 - ig \sinh(\theta - \theta')} . \quad (4.23)$$

Recall the value of the parameter  $g$  and  $m$  in terms of the 't Hooft coupling  $\lambda$  from (4.2).

On the basis of integrability, the knowledge of the two-body S-matrix is enough to immediately write down the Bethe equations:

$$e^{iLm \sinh \theta_j} = \prod_{k \neq j} \frac{1 - ig \sinh(\theta_j - \theta_k)}{1 + ig \sinh(\theta_j - \theta_k)} . \quad (4.24)$$

Energy and momentum are given by

$$E = m \sum_j \cosh \theta_j \quad , \quad P = m \sum_j \sinh \theta_j . \quad (4.25)$$

It is interesting to compare our result to the semiclassical spectrum of the AdS string. Because of the fermionic nature of the  $\mathfrak{su}(1|1)$  subsector, there are no classical spinning-string solutions, but one can study states with very low momentum  $p \sim 1/L$  (BMN states), which correspond to the lowest string modes.  $1/L$  corrections to the energies of these states were calculated explicitly in [18] and can be encoded in a compact form in a set of Bethe equations [20] with the S-matrix<sup>6</sup>

$$S(p, p') = \exp \left[ \frac{i}{2} (p(\varepsilon(p') - \varepsilon(p)p')) \right] , \quad (4.26)$$

where<sup>7</sup>

$$\varepsilon(p) = \sqrt{1 + \frac{\lambda}{4\pi^2} p^2} . \quad (4.27)$$

The same Bethe equations were shown to arise from quantization of the free fermions in the light-cone gauge [40]. We now compare the S-matrix (4.26) to (4.23). For easier comparison, we give up our vector notation ( $\vec{p}$ ) and use  $p$  for the momentum and  $\varepsilon$  for the energy. Furthermore we have to take into account that we considered the AAF model with rescaled time coordinate, i.e. we now compensate for this by rescaling the energy by  $\frac{\sqrt{\lambda}}{2\pi}$ , so the dispersion relation becomes precisely (4.27) and the S-matrix is

$$S(p, p') = \frac{1 + \frac{i}{4} (p\varepsilon(p') - \varepsilon(p)p')}{1 - \frac{i}{4} (p\varepsilon(p') - \varepsilon(p)p')} . \quad (4.28)$$

It is immediately obvious that the S-matrices coincide in the low momentum approximation, but deviate from each other at large momenta.

We should warn the reader that the Bethe equations (4.24) cannot be used to describe the quantum spectrum of the string in  $\text{AdS}_5 \times S^5$ . Quantum corrections are obviously different for the quantum field theory defined by (4.3) and the full string sigma-model. It is still interesting to note that quantum corrections in (4.24) (deviations from the classical phase shift (4.26)) are not of the form anticipated for the quantum string in [19, 22]. We would also like to stress that (4.24) are bare Bethe equations, they have solutions with negative energies, and the true ground state is the solution with the Dirac sea filled.

We now turn to the discussion of the physical ground state of the AAF model. Recall that for technical reasons we quantized the theory in the pseudo vacuum  $|0\rangle$ , cf. (4.8). The physical vacuum  $|\Omega\rangle$  is obtained from the pseudo-vacuum by exciting all negative modes, symbolically

$$|\Omega\rangle = \prod_{p_1} b^\dagger(p_1) |0\rangle . \quad (4.29)$$

This state corresponds to a non-trivial solution of the bare Bethe equations. Recall that the  $b^\dagger$ -particles are excitations with complex rapidities  $\theta_j = i\pi - \alpha_j$ . Hence, the state

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<sup>6</sup>To compare to [20] one has to take into account that the R-charge denoted there by  $J$  is what we call  $L$  here. If one identifies  $L + (\text{number of Bethe roots})/2$  with the length, as is done in [20], then the S-matrix effectively acquires an extra factor of  $\exp[i(p' - p)/2]$ , cf. eq. (4.31) in [20].

<sup>7</sup>To be more precise, a lattice dispersion relation with  $p^2$  replaced by  $4\sin^2(p/2)$  was postulated in [20]. The difference, however, is  $O(1/L^2)$  for the BMN states.

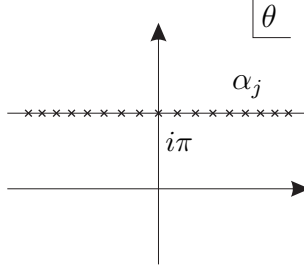


Figure 9: **Bethe roots for physical vacuum.** With respect to the pseudo-vacuum  $|0\rangle$ , which underlies the “bare” Bethe equations, the physical vacuum  $|\Omega\rangle$  is a highly excited state. It corresponds to an  $i\pi$ -line completely filled with Bethe roots  $\alpha_j$ .

$|\Omega\rangle$  is described by placing Bethe roots as densely as possible along the  $i\pi$ -line. This is depicted in Fig. 9. One can construct the vacuum solution of the Bethe equations in the thermodynamic ( $L \rightarrow \infty$ ) limit, find the spectrum of physical excitations and their scattering S-matrix by solving certain integral equations. The details of the derivation can be found in the original literature on the massive Thirring model [42, 43], in the review [32] or in the monograph [15]. For our model, the density of roots in the vacuum  $\rho_\Omega(\alpha)$  satisfies the following equation

$$m \cosh \alpha = 2\pi \rho_\Omega(\alpha) + \int_{-\infty}^{+\infty} d\bar{\alpha} \rho_\Omega(\bar{\alpha}) \frac{2g \cosh(\alpha - \bar{\alpha})}{1 + g^2 \sinh^2(\alpha - \bar{\alpha})}. \quad (4.30)$$

The energy density of the physical vacuum is given by

$$\frac{E}{L} = -m \int d\alpha \rho_\Omega(\alpha) \cosh \alpha. \quad (4.31)$$

The physical excitations are obtained from the vacuum configuration of Fig. 9 by inserting further Bethe roots outside the  $i\pi$ -line or by making holes in the Dirac sea. The energy of those physical configurations is then measured with respect to the ground state energy (4.31).

Let us end this section with discussing the solutions of (4.30) in various regimes of the coupling  $g = \frac{\pi}{2\sqrt{\lambda}}$ . In the attractive regime,  $g < 0$ , we find that  $\rho_\Omega$  is determined only up to a constant. This is because a constant shift of the density:  $\rho_\Omega(\alpha) \rightarrow \rho_\Omega(\alpha) + C$  does not affect the left hand side of the equation. Choosing this constant arbitrarily large one can make the energy (4.31) arbitrarily negative, which leads to a vacuum instability for any attractive coupling constant. This behavior is similar to what happens in the massive Thirring model at infinite coupling, which corresponds there to the phase transition point. In the AAF model, however, the instability is present for all values of  $g < 0$ .

In the repulsive regime,  $g > 0$ , there is no such problem. We solve (4.30) by Fourier transformation. However, we are required to introduce a rapidity cut-off  $|\alpha| \leq \Lambda$  in order to regularize the Fourier transform of  $\cosh \alpha$ . Then the root density of  $|\Omega\rangle$  can be written as the Fourier integral

$$\rho_\Omega(\alpha) = \frac{m}{16\pi^2} \int_{-\infty}^{\infty} da e^{-ia\alpha} \frac{\cosh \frac{\pi}{2}a}{\cosh \frac{\mu}{2}a \cosh \frac{\pi-\mu}{2}a} \left[ \frac{e^{(1+ia)\Lambda}}{1+ia} + \frac{e^{(1-ia)\Lambda}}{1-ia} \right]. \quad (4.32)$$

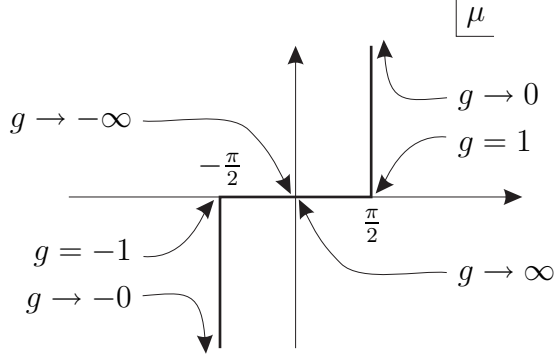


Figure 10: **Coupling constant in AAF model.** It is convenient to parameterize the coupling constant  $g$  by a complex variable  $\mu$  in the way shown here. This is a solution of  $\sin \mu = 1/g$ .

Here  $\mu = \arcsin(1/g)$ , where we use the branches of  $\arcsin$  as depicted in Fig. 10. Notice that  $\mu$  is real for strong coupling  $g > 1$ , but complex for weak coupling  $0 < g \leq 1$ . The integral (4.32) can be computed by contour integration. In the limit  $\Lambda \rightarrow \infty$ , the dominant contribution to the integral stems from the poles at  $a = \pm i \frac{\pi}{\pi - \mu}$  in the strong coupling regime and from the poles at  $a = \pm i \frac{\pi}{\pi - \mu^*}$  and  $a = \pm i \frac{\pi}{\pi - \mu}$  in the weak coupling regime. For  $g > 1$  we have

$$\rho_{\Omega}(\alpha) = \frac{m}{2\pi} e^{-\frac{\pi}{\pi - \mu} \Lambda} \frac{1}{\mu} \cot \frac{\pi^2}{2(\mu - \pi)} \cosh \frac{\pi \alpha}{\pi - \mu}, \quad (4.33)$$

for  $0 < g \leq 1$  we have (4.33) plus its complex conjugate. In the strong coupling regime, we can introduce a renormalized mass

$$m_R \sim m e^{-\frac{\pi}{\pi - \mu} \Lambda} \quad (4.34)$$

in order to get a finite result when the cut-off is removed. In the weak coupling regime this is not possible as the two terms in  $\rho_{\Omega}(\alpha)$  have a different dependence on  $\Lambda$ . Hence, the model is non-renormalizable for  $0 < g = \frac{\pi}{2\sqrt{\lambda}} \leq 1$ . It is instructive to express (4.34) in terms of the original coupling  $g$ . In the limit  $g \rightarrow 0$ , we find that the anomalous dimension of the mass  $m$  behaves as

$$\gamma_m = \frac{\pi}{\pi - \mu} \sim \frac{i}{\ln g}. \quad (4.35)$$

The approach of the anomalous dimension to zero as  $g \rightarrow 0$  is not analytic. We see that the breakdown of renormalizability is a non-perturbative effect.

In the strong repulsive regime ( $g > 1$ ), the AAF model seems to be a well defined quantum theory. Unfortunately, additional complications arise in this case, because one can further diminish the vacuum energy by inserting  $n$ -strings around the  $i\pi$ -line. We thus expect that the true vacuum is a condensate of  $n$ -strings, where  $n$  can depend on  $g$ , like in the strongly repulsive Thirring model [44].

## 5 Faddeev-Reshetikhin model

In this section we study string theory on  $S^3 \times R$ . We should probably explain what we mean by that because  $S^3 \times R$  is not a string background. There are no problems with classical strings, in fact  $S^3 \times R$  can be regarded as a subspace of  $\text{AdS}_5 \times S^5$ , but quantization leads to UV divergences and non-zero beta-function. Although the resulting model cannot be interpreted as string theory, it is an interesting example of two-dimensional integrable field theory first considered in [30].

The string action in the conformal gauge is<sup>8</sup>

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[ \frac{1}{2} \text{tr} j_a^2 + (\partial_a X^0)^2 \right], \quad (5.1)$$

where  $X^0$  is the time coordinate and

$$j_a = g^{-1} \partial_a g. \quad (5.2)$$

Here  $g$  is a group element of  $SU(2)$  that parameterizes an embedding of the string world-sheet in  $S^3$ . The current satisfies the following equations of motion:

$$\begin{aligned} \partial_a j^a &= 0, \\ \partial_a j_b - \partial_b j_a + [j_a, j_b] &= 0. \end{aligned} \quad (5.3)$$

In addition we should impose the Virasoro constraints:

$$\text{tr} j_{\pm}^2 = -2 (\partial_{\pm} X^0)^2, \quad (5.4)$$

where  $\partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}$  are derivatives with respect to the light-cone coordinates  $\sigma^{\pm} = (\tau \pm \sigma)/2$ . Accordingly,  $j_{\pm}$  are the light-cone components of the  $su(2)$  current.

The standard way to proceed would be to quantize the sigma-model and then impose the Virasoro constraints in the weak sense – as subsidiary conditions on the physical states. This procedure is inconsistent for the model at hand, for the reasons explained above. It is also not likely to work for the full string sigma-model in  $\text{AdS}_5 \times S^5$  because of the well-known problems with the conformal gauge for the Green-Schwarz superstring. Another (in fact, no less standard) approach to string quantization is to fix the gauge completely, solve the Virasoro constraints (eliminate transverse degrees of freedom), and only then quantize. We thus choose the temporal gauge:

$$X^0 = \kappa \tau, \quad (5.5)$$

such that the Virasoro constraints become

$$\text{tr} j_{\pm}^2 = -2\kappa^2, \quad (5.6)$$

and are solved by

$$j_{\pm} = i\kappa \vec{S}_{\pm} \cdot \vec{\sigma}, \quad (5.7)$$

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<sup>8</sup>The string tension of the AdS string is  $\sqrt{\lambda}/2\pi$ .

where  $\vec{S}_+$  and  $\vec{S}_-$  are three-dimensional vectors of unit norm:  $\vec{S}_\pm^2 = 1$ . They satisfy the equations of motion

$$\begin{aligned}\partial_+ S_-^i + \kappa \varepsilon^{ijk} S_-^j S_+^k &= 0, \\ \partial_- S_+^i + \kappa \varepsilon^{ijk} S_-^j S_+^k &= 0.\end{aligned}\tag{5.8}$$

Now we face an immediate problem: What Hamiltonian should be quantized? The original Hamiltonian, canonical conjugate to  $\tau$  in (5.1) is set to zero by the Virasoro constraints. The way out is to use another Hamiltonian and another Poisson structure, proposed in [30]. They are more natural from the point of view of integrability, and are potentially relevant in the AdS/CFT correspondence [5, 45]. The action in the path integral for the evolution operator associated with the Hamiltonian of [30] is

$$S = \int d^2x \left[ C_+(\vec{S}_-) + C_-(\vec{S}_+) - \frac{\kappa}{2} \vec{S}_+ \cdot \vec{S}_- \right], \tag{5.9}$$

where  $C_\pm$  are Wess-Zumino terms as defined in (3.3). This will be our starting point. It is easy to check that the equations of motion (5.8) follow from variation of this action.

To develop perturbation theory for the FR model, we will perform the same change of variables as in sec. 3:

$$\phi_\pm = \frac{S_\pm^1 + iS_\pm^2}{\sqrt{2 + 2S_\pm^3}}, \quad S_\pm^3 = 1 - 2|\phi_\pm|^2. \tag{5.10}$$

Upon this change of variables, the Wess-Zumino term becomes a canonically normalized first-order action for  $\phi_+$ ,  $\phi_-$ :

$$\begin{aligned}S = \int d^2x & \left[ \frac{i}{2} (\phi_+^* \partial_- \phi_+ - \partial_- \phi_+^* \phi_+ + \phi_-^* \partial_+ \phi_- - \partial_+ \phi_-^* \phi_-) \right. \\ & - \kappa \sqrt{(1 - |\phi_+|^2)(1 - |\phi_-|^2)} (\phi_+^* \phi_- + \phi_-^* \phi_+) + \kappa (|\phi_+|^2 + |\phi_-|^2) \\ & \left. - 2\kappa |\phi_+|^2 |\phi_-|^2 \right].\end{aligned}\tag{5.11}$$

This action can be cast into a very concise form if we combine  $\phi_+$  and  $\phi_-$  into a two-component commuting spinor:

$$\phi = \begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix}. \tag{5.12}$$

Then (5.11) becomes a Dirac-like action<sup>9</sup>

$$S = \int d^2x \left( i\bar{\phi} \not{D} \phi - m\bar{\phi} \phi - g \bar{\phi} \gamma^\mu \phi \bar{\phi} \gamma_\mu \phi + O(\phi^6) \right). \tag{5.13}$$

The covariant derivative contains a field-dependent chemical potential:

$$D_0 = \partial_0 - im - ig \bar{\phi} \phi, \quad D_1 = \partial_1. \tag{5.14}$$

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<sup>9</sup>To make power-counting possible we introduce the mass  $m = \kappa$  and the coupling constant  $g = \kappa/2$ .

This action describes a single charged particle and its anti-particle with the dispersion relations

$$\varepsilon = \sqrt{p^2 + m^2} - m \quad (\text{particle}) , \quad (5.15)$$

and

$$\varepsilon = \sqrt{p^2 + m^2} + m \quad (\text{anti-particle}) . \quad (5.16)$$

This first of these equations is the BMN formula [46]. The mass gap for the particles is offset to zero by the chemical potential. The energy of anti-particles is shifted in the opposite direction, so that anti-particles decouple at low energies and momenta. The low-energy effective theory can be constructed by defining big and small components of the spinor:

$$\varphi = \frac{1 + \gamma^0}{2} \phi, \quad \chi = \frac{1 - \gamma^0}{2} \phi . \quad (5.17)$$

The Lagrangian in (5.13) takes the form

$$\mathcal{L} = i \varphi^* \partial_0 \varphi + i \chi^* \partial_0 \chi + i \varphi^* \partial_1 \chi + i \chi^* \partial_1 \varphi + 2m |\chi|^2 + g (\varphi^{*2} \chi^2 + \chi^{*2} \varphi - |\chi|^4) . \quad (5.18)$$

Integrating out  $\chi$  we arrive at

$$\mathcal{L} = i \varphi^* \partial_0 \varphi - \frac{1}{2m} |\partial_1 \varphi|^2 - \frac{g}{4m^2} [(\varphi^* \partial_1 \varphi)^2 + (\partial_1 \varphi^* \varphi)^2] + \dots , \quad (5.19)$$

which is the same as (3.20). This is another way to see that the Landau-Lifshitz model is the low-energy effective theory for strings on  $S^3 \times R$ .

Returning to the sigma-model, we can now quantize (5.13) in the “wrong” vacuum, in which all anti-particle states are empty. Then the S-matrix can be computed by summing the bubble diagrams. If the pole prescription is non-relativistic, the field-independent part of the chemical potential in (5.14) can be eliminated by a shift of the integration variable  $k^0$ , and we can use the relativistic dispersion relation instead of the BMN formulas (5.15), (5.16). Calculation of the loop integral for the  $pp' \rightarrow pp'$  scattering (the details can be found in appendix C) gives:

$$\text{Loop} = \frac{(\not{p}' + m) \otimes (\not{p} + m) + (\not{p} + m) \otimes (\not{p}' + m)}{8m^2 \sinh(\theta - \theta')} . \quad (5.20)$$

The S-matrix is<sup>10</sup>

$$\begin{aligned}
\langle k k' | \hat{S} | p p' \rangle &= 4p_0 p'_0 (2\pi)^2 \delta_+(k, k', p, p') \\
&+ (2\pi)^2 \delta^{(2)}(k + k' - p - p') \frac{ig}{2} \langle \text{out} | (\gamma^0 \otimes 1 + 1 \otimes \gamma^0 - 2\gamma^\mu \otimes \gamma_\mu) \\
&\times \sum_{n=0}^{\infty} \left[ \frac{(\not{p}' + m) \otimes (\not{p} + m) + (\not{p} + m) \otimes (\not{p}' + m)}{8m^2 \sinh(\theta - \theta')} \right. \\
&\quad \left. \times ig (\gamma^0 \otimes 1 + 1 \otimes \gamma^0 - 2\gamma^\alpha \otimes \gamma_\alpha) \right]^n | \text{in} \rangle, \tag{5.21}
\end{aligned}$$

where by  $| \text{in} \rangle$  and  $\langle \text{out} |$  we denote bi-spinors

$$\begin{aligned}
| \text{in} \rangle &= u(p) \otimes u(p') + u(p') \otimes u(p), \\
\langle \text{out} | &= \bar{u}(p) \otimes \bar{u}(p') + \bar{u}(p') \otimes \bar{u}(p). \tag{5.22}
\end{aligned}$$

Observing that

$$\not{p} + m = u(p) \bar{u}(p), \tag{5.23}$$

and taking into account the symmetry of the vertex, we can replace

$$(\not{p}' + m) \otimes (\not{p} + m) + (\not{p} + m) \otimes (\not{p}' + m) \rightarrow \frac{1}{2} | \text{in} \rangle \langle \text{out} |. \tag{5.24}$$

Then the S-matrix becomes

$$S(\theta, \theta') = 1 + 2 \sum_{n=1}^{\infty} \left[ \frac{ig \langle \text{out} | (\gamma^0 \otimes 1 + 1 \otimes \gamma^0 - 2\gamma^\alpha \otimes \gamma_\alpha) | \text{in} \rangle}{16m^2 \sinh(\theta - \theta')} \right]^n. \tag{5.25}$$

Plugging the explicit form of the wave functions  $\bar{u}(p)$  and  $u(p)$  into the matrix element of the vertex, we finally get:

$$S(\theta, \theta') = \frac{1 + ig \left( \frac{\cosh \frac{\theta + \theta'}{2}}{\sinh \frac{\theta - \theta'}{2}} - \coth \frac{\theta - \theta'}{2} \right)}{1 - ig \left( \frac{\cosh \frac{\theta + \theta'}{2}}{\sinh \frac{\theta - \theta'}{2}} - \coth \frac{\theta - \theta'}{2} \right)}. \tag{5.26}$$

It is easy to check that in the limit of small momenta the S-matrix of the LL model (3.30) is recovered.

The S-matrix of the FR model is not Lorentz invariant. This is not surprising because the Virasoro constraints explicitly break the Lorentz symmetry of the originally Lorentz-invariant chiral field. Because of the lack of Lorentz invariance the S-matrix is not a

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<sup>10</sup>We change our conventions for the wave functions of external states. Here

$$u(p) = \sqrt{m} \begin{pmatrix} e^{-\theta/2} \\ e^{\theta/2} \end{pmatrix}$$

describes pseudo-particles with both positive and negative energies. The spinors are normalized as  $\bar{u}(p)u(p) = 2m$ .



function of  $\theta - \theta'$ , which makes rapidity parameterization not the most convenient one. There is another parameterization of energies and momenta which is much more useful in this particular case:

$$\begin{aligned}\frac{\varepsilon}{m} &= \cosh \theta = \frac{x^2 + 1}{x^2 - 1}, \\ \frac{p}{m} &= \sinh \theta = \frac{2x}{x^2 - 1}.\end{aligned}\tag{5.27}$$

The S-matrix takes an extremely simple form in these variables:

$$S(x, x') = \frac{x - x' - 2ig}{x - x' + 2ig}.\tag{5.28}$$

We can now write down the Bethe equations:

$$\exp\left(\frac{2imLx_j}{x_j^2 - 1}\right) = \prod_{k \neq j} \frac{x_j - x_k + 2ig}{x_j - x_k - 2ig}.\tag{5.29}$$

The states with Bethe roots in the interval  $-1 < x_j < 1$  carry negative energy. In quantum theory all of the negative-energy levels should be filled. The repulsive ( $g < 0$ ) and the attractive ( $g > 0$ ) cases are very different in this respect. In the repulsive case, the roots cannot form strings and the ground-state density in the thermodynamic limit is determined by the following integral equation

$$m \frac{1 + x^2}{(1 - x^2)^2} = \pi \rho(x) - 2|g| \int_{-1}^1 \frac{dy \rho(y)}{(x - y)^2 + 4g^2}.\tag{5.30}$$

The singularities at  $x = \pm 1$  require regularization and will lead to renormalization of the bare parameters. It would be interesting to solve this equation and to perform the renormalization explicitly.

The situation is more complicated in the attractive regime of  $g > 0$ . In this case, both positive- and negative-energy roots can form strings with  $\Delta x = 2ig$ . A simple calculation shows that the energy of the  $n$ -string with the centre of mass between  $-1$  and  $1$  decreases with  $n$  (becomes more and more negative) up to  $n \sim 1/2g$ . We thus expect that the ground state is a condensate of  $n$ -strings with very large  $n \sim 1/g$ . This is in a qualitative agreement with the results of Faddeev and Reshetikhin for the lattice-regularized model [30]. The vacuum of the lattice model is a condensate of  $2S$ -strings with  $S \rightarrow \infty$ . We will not attempt to solve the Bethe equations (5.29) here, but we expect that the renormalized solution describes the quantized principal chiral field [31], as it is the case for the lattice model of [30].

## 6 Conclusions

We hope that our calculations illustrate several features of the Bethe ansatz that can be useful in solving string theory in  $\text{AdS}_5 \times S^5$ . The consistency of the approach we

used fully relies on quantum integrability which reduces the many-particle problem to superposition of two-body interactions. This allowed us to avoid complexities associated with highly non-linear interactions in the sigma-models. Let us take the LL model as an example. Its Lagrangian contains infinitely many non-linear terms, but we needed only the quartic vertices to derive the Bethe equations and thus to reconstruct the full spectrum. In other words we managed to solve the model by analyzing just the small fluctuations around the "north pole"  $\vec{n} = (0, 0, 1)$ . To illustrate this point, imagine that the target space of the LL sigma-model (the sphere  $S^2$ ) is deformed near the south pole, for instance that the constraint  $\vec{n}^2 = 1$  is replaced by  $\vec{n}^2 = f(n_3)$ . If  $f(1-\varepsilon) = 1 + \mathcal{O}(\varepsilon^3)$ , the quartic vertices of perturbation theory around  $\vec{n} = (0, 0, 1)$  are still the same and we will get the same two-body S-matrix. The complete non-perturbative spectrum of the deformed model, however, is totally different. The full spectrum is certainly not determined by the two-body S-matrix, because the deformation completely destroys the integrability and multi-particle interactions do not factorize any more.

To further illustrate the power of integrability, we note that the Bethe ansatz completely determines the spectrum of a system in a finite volume, which is particularly important for applications to string theory, but we did not use periodic boundary conditions anywhere in deriving Bethe equations. In fact, the S-matrix is only defined in the infinite volume where the notion of asymptotic states makes sense.

Although Bethe ansatz completely determines the spectrum, it may happen that physically interesting states with low energies are described by rather complicated solutions of the Bethe equations. The ground state for a relativistic system typically is a Dirac sea that contains a non-trivial distribution of infinitely many roots<sup>11</sup>. Usually one can find the ground-state distribution in a closed form only in the thermodynamic limit. It is interesting to note here that anti-particle-like states with negative energies seem to also arise on the gauge-theory side of the AdS/CFT correspondence. The all-loop Bethe equations for the spectrum of local operators in  $\mathcal{N} = 4$  SYM [8] do have negative-energy solutions at the non-perturbative level [26, 17]. An interpretation and a physical significance of these states remains unclear to us. In particular, in [17] the negative-energy states were projected out rather than filled. We believe that these states are direct counterparts of the anti-particles in the sigma-model. Understanding their role in the AdS/CFT correspondence is a very interesting and important problem.

**Note added.** A complementary approach to the quantum Bethe ansatz in the string-like sigma-models, which is based on the known exact physical S-matrices [16], was developed in a parallel publication [47].

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<sup>11</sup>We would like to thank F. Smirnov for the discussion of this point.

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## A Computational details for LL model

### A.1 Loop integrals

A generic higher loop diagram in the LL model consists of a chain of bubbles as depicted in Fig. 4. In this appendix we compute the “bubble propagators”  $I_r$ . More specifically, we compute the momentum space representation of two parallel propagators  $D(t, x)$ , cf. (3.16), with two inflowing on-shell momenta  $p$  and  $p'$  ( $p > p'$ ). We need to consider the cases with  $r = 0, 1, 2$  pairs of derivatives acting onto the two propagators. In the case without derivatives we have

$$\begin{aligned}
 I_0(p, p') &= \text{Diagram: A bubble diagram with two vertices. The left vertex has two incoming dashed lines labeled $p$ and $p'$. The right vertex has two outgoing dashed lines labeled $p$ and $p'$. The bubble consists of two curved lines connecting the vertices. The top curved line is labeled $q$ and the bottom curved line is labeled $p + p' - q$.} \\
 &= \int dt dx (D(x, t))^2 e^{i(p^2 + p'^2)t - i(p + p')x} \\
 &= \int \frac{d\omega dq}{(2\pi)^2} \frac{i^2}{[\omega - q^2 + i\epsilon][p^2 + p'^2 - \omega - (p + p' - q)^2 + i\epsilon]} .
 \end{aligned} \tag{A.1}$$

First perform the energy integral over  $\omega$  by contour integration. The integrand has one pole in the upper and one pole in the lower half plane. Closing the contour in either half plane leads to

$$I_0(p, p') = -\frac{i}{2} \int \frac{dq}{2\pi} \frac{1}{(q - \frac{1}{2}p - \frac{1}{2}p')^2 - \frac{1}{4}(p - p')^2 - i\epsilon} . \tag{A.2}$$

Shift the integration variable by  $q \rightarrow q + \frac{1}{2}p + \frac{1}{2}p'$  and perform the momentum integration also by contour integration. One finds

$$I_0(p, p') = \frac{1}{2(p - p')} . \tag{A.3}$$

Loops with derivatives, see (A.8) and (A.9), are similar to the above. Every pair of derivatives introduces a factor of  $-k(p - k)$  into the numerator. The energy integral is unchanged. However, the momentum integral is now divergent. For the evaluation of these integrals we invoke dimensional regularization. This has the same effect as adding appropriate counterterms to the action. The integrals become finite and their values are

$$I_1(p, p') = \frac{-pp'}{2(p - p')} , \tag{A.4}$$

$$I_2(p, p') = \frac{(pp')^2}{2(p - p')} . \tag{A.5}$$

## A.2 Summing all Feynman diagrams

We compute

$$\langle k k' | \hat{S} | p p' \rangle \Big|_{g^n} = \langle k k' | \frac{1}{n!} \left[ \left( -\frac{ig}{2} \right) \int dx dt [(\varphi^* \partial_x \varphi)^2 + (\partial_x \varphi^* \varphi)^2] \right]^n | p p' \rangle . \quad (\text{A.6})$$

The vertices are to be connected as in Fig. 4. Any other diagram will be zero. There are  $2^{n+1}n!$  different ways of connecting  $n$  four-vertices among themselves and to the external legs in the way of Fig. 4. Furthermore we need to take into account that we have two different vertices. This amounts to placing derivatives onto the legs of the vertex, either onto the pair of in-going or on the pair of out-going lines. This leads to  $2^n$  different diagrams.

A generic diagram consists of a chain of loops. If the derivatives hit the external legs we get a factor of  $-pp'$ . Derivatives on internal lines lead to three different kinds of loops:

$$I_0(p, p') = \text{diagram with two vertices and two internal lines, no external leg derivatives} \quad (\text{A.7})$$

$$I_1(p, p') = \text{diagram with two vertices and two internal lines, one external leg derivative} = \text{diagram with two vertices and two internal lines, one internal line derivative} \quad (\text{A.8})$$

$$I_2(p, p') = \text{diagram with two vertices and two internal lines, two external leg derivatives} \quad (\text{A.9})$$

The corresponding loop integrals are evaluated in the previous subsection. Counting how many of the  $2^n$  diagrams lead to a given expression

$$(-pp')^{\#1} I_0^{\#2}(p, p') I_1^{\#3}(p, p') I_2^{\#4}(p, p') \quad (\text{A.10})$$

is not completely trivial in general. This counting, however, becomes unnecessary as we find

$$I_1(p, p') = (-pp') I_0(p, p') \quad , \quad I_2(p, p') = (-pp')^2 I_0(p, p') . \quad (\text{A.11})$$

This means that every diagram, no matter how the derivatives are distributed, will contribute

$$(-pp')^n I_0^{n-1}(p, p') = \frac{(-pp')^n}{(2(p-p'))^{n-1}} . \quad (\text{A.12})$$

Multiplying all factors together

$$\begin{aligned} \langle k k' | \hat{S} | p p' \rangle \Big|_{g^n} &= 2^{n+1} n! \cdot 2^n \cdot \frac{1}{n!} \cdot \left( -\frac{ig}{2} \right)^n \cdot \frac{(-pp')^n}{(2(p-p'))^{n-1}} \cdot \frac{1}{\lambda'(p-p')} \delta_+(p, p', k, k') \\ &= 2 (ig)^n \left( \frac{pp'}{p-p'} \right)^n \delta_+(p, p', k, k') . \end{aligned} \quad (\text{A.13})$$

For  $n = 1$  we recover the tree-level result (3.27) computed in the main text.

### A.3 Two-particle Green's function

Here we compute

$$\langle 0 | T \mathcal{O}(t, x) \mathcal{O}(0, 0) | 0 \rangle = \frac{i}{4} \int \frac{d\Omega dP}{(2\pi)^2} e^{-i\Omega t + iPx} G(\Omega, P) , \quad (\text{A.14})$$

where  $\mathcal{O}$  is defined in (3.39). For that we need to compute the loop diagrams  $I_0$ ,  $I_1$  and  $I_2$  off-shell:

$$I_0(\Omega, P) = - \int \frac{d\omega dk}{(2\pi)^2} \frac{1}{(\omega - k^2)[\Omega - \omega - (P - k)^2]} = \int_{-\infty}^{+\infty} \frac{dk}{2\pi i} \frac{1}{k^2 + (P - k)^2 - \Omega} . \quad (\text{A.15})$$

The answer is the same (A.3)-(A.5), where  $p, p'$  are defined as the roots of the denominator in the integrand:

$$\left\{ \begin{array}{c} p \\ p' \end{array} \right\} = \frac{P}{2} \pm \frac{i}{2} \sqrt{P^2 - 2\Omega} . \quad (\text{A.16})$$

In particular,

$$I_0(\Omega, P) = \frac{1}{2i\sqrt{P^2 - 2\Omega}} . \quad (\text{A.17})$$

Summing the bubble diagrams as before we find:

$$G(\Omega, P) = \frac{4(p p')^2}{2pp' + \frac{i}{I_0(\Omega, P)}} , \quad (\text{A.18})$$

which gives (3.40).

## B Computational details for AAF model

### B.1 Spinors

In the AAF model we work with the following two-component spinors:

$$u(p_1) = \begin{pmatrix} \sin \eta(p_1) \\ \cos \eta(p_1) \end{pmatrix} , \quad v(p_1) = \begin{pmatrix} \cos \eta(p_1) \\ -\sin \eta(p_1) \end{pmatrix} \quad (\text{B.1})$$

and their Dirac conjugates  $\bar{u} = u^\dagger \gamma^0$  and  $\bar{v} = v^\dagger \gamma^0$ . The angle  $\eta(p_1)$  is defined through the relation  $\cot 2\eta(p_1) = \frac{p_1}{m}$ . We use the following explicit representation of the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^3 = \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \quad (\text{B.2})$$

These spinors obey the equations of motion

$$(\not{p} - m)u(p_1) = 0 , \quad (\not{p} + m)v(-p_1) = 0 , \quad (\text{B.3})$$

where  $p_0 = \sqrt{p_1^2 + m^2}$ . They satisfy the completeness relations

$$u(p_1)\bar{u}(p_1) = \frac{1}{2p_0}(\not{p} + m) \quad , \quad v(p_1)\bar{v}(p_1) = \frac{1}{2p_0}(\not{p} - m) \quad (\text{B.4})$$

and the orthogonality relations

$$u^\dagger(p_1)u(p_1) = v^\dagger(p_1)v(p_1) = 1 \quad , \quad u^\dagger(p_1)v(p_1) = v^\dagger(p_1)u(p_1) = 0 \quad (\text{B.5})$$

and

$$\bar{u}(p_1)u(p_1) = -\bar{v}(p_1)v(p_1) = \frac{m}{p_0} \quad , \quad u^\dagger(p_1)v(p_1) = v^\dagger(p_1)u(p_1) = \frac{p_1}{p_0} . \quad (\text{B.6})$$

## B.2 Loop integrals

We compute the “bubble propagators”  $I_r$  in the AAF model. This is essentially a repetition of the computation in the LL model (App. A.1) where now the fermion propagator (4.9) is used. Defining  $\vec{P} = \vec{p} + \vec{p}'$  we find:

$$I_0(\vec{p}, \vec{p}') = \text{Diagram} \quad (\text{B.7})$$

$$= \int \frac{d^2 q}{(2\pi)^2} D(\vec{q}) \otimes D(\vec{p} + \vec{p}' - \vec{q})$$

$$= \frac{\vec{p}' \times \vec{p}}{4\vec{P}^2} \gamma^\alpha \otimes \gamma_\alpha ,$$

$$I_1(\vec{p}, \vec{p}') = \text{Diagram} = \text{Diagram} \quad (\text{B.8})$$

$$= \int \frac{d^2 q}{(2\pi)^2} (-\vec{q} \times (\vec{p} - \vec{q})) D(\vec{q}) \otimes D(\vec{p} + \vec{p}' - \vec{q})$$

$$= \frac{\vec{p}' \times \vec{p}}{4\vec{P}^2} \cdot \left[ \frac{1}{2} (\not{P}\gamma^3 \otimes \not{P} - \not{P} \otimes \not{P}\gamma^3) + m (\not{P}\gamma^3 \otimes \mathbf{1} - \mathbf{1} \otimes \not{P}\gamma^3) \right] ,$$

$$\begin{aligned}
I_2(\vec{p}, \vec{p}') &= \\
&= \int \frac{d^2 q}{(2\pi)^2} (-\vec{q} \times (\vec{p} - \vec{q}))^2 D(\vec{q}) \otimes D(\vec{p} + \vec{p}' - \vec{q}) \\
&= \frac{\vec{p}' \times \vec{p}}{4\vec{P}^2} \cdot \left[ (\vec{p}' \times \vec{p})^2 \gamma^\alpha \otimes \gamma_\alpha + m^2 \not{P} \otimes \not{P} \right. \\
&\quad \left. + \frac{m}{2} \vec{P}^2 (\not{P} \otimes \mathbf{1} - \mathbf{1} \otimes \not{P}) + m^2 \vec{P}^2 \mathbf{1} \otimes \mathbf{1} \right].
\end{aligned} \tag{B.9}$$

In the computation we made use of

$$\int \frac{d^2 q}{(2\pi)^2} \frac{1}{(q^2 + \Delta)^2} = 0, \tag{B.10}$$

$$\int \frac{d^2 q}{(2\pi)^2} \frac{q^2}{(q^2 + \Delta)^2} = \frac{1}{4} \theta(\Delta), \tag{B.11}$$

$$\int \frac{d^2 q}{(2\pi)^2} \frac{q^4}{(q^2 + \Delta)^2} = -\frac{1}{2} \Delta \theta(\Delta), \tag{B.12}$$

for  $\Delta \in R$ . These formulas can be obtained from taking derivatives of the more general integral with numerator  $e^{i\vec{p}\cdot\vec{\alpha}}$ . This integral itself can be evaluated by contour integration.

### B.3 Summing all Feynman diagrams

We compute

$$\langle k k' | \hat{S} | p p' \rangle \Big|_{g^n} = \langle k k' | \frac{1}{n!} \left[ \left( -\frac{ig}{m^2} \right) \int dx dt \varepsilon^{\alpha\beta} [\bar{\psi} \partial_\alpha \psi \bar{\psi} \gamma^3 \partial_\beta \psi - \partial_\alpha \bar{\psi} \psi \partial_\beta \bar{\psi} \gamma^3 \psi] \right]^n | p p' \rangle. \tag{B.13}$$

For the reasons discussed in the main text only bubble diagrams contribute. As opposed to the LL model, here the chain of bubbles is made of two distinguished strands which connect one in-going particle with one out-going particle each. I.e. there are two cases (Fig. 11):  $\vec{p}$  connected to  $\vec{k}$  and  $\vec{p}'$  connected to  $\vec{k}'$  or  $\vec{p}$  connected to  $\vec{k}'$  and  $\vec{p}'$  connected to  $\vec{k}$ . The strands describe the flow of the spinor indices. At the vertex they hit onto the unit matrix or  $\gamma^3$ . The two different possibilities can be taken into account by writing the factor

$$\mathbb{P} = \mathbf{1} \otimes \gamma^3 - \gamma^3 \otimes \mathbf{1} \quad , \quad \left( \frac{1}{2} \mathbb{P} \right)^3 = \frac{1}{2} \mathbb{P} \tag{B.14}$$

at each vertex in Fig. 11. If there was only the first vertex and if we neglect the derivatives for a moment, then all diagrams at order  $g^n$  are given by

$$(\bar{u}(k'_1) \otimes \bar{u}(k_1) - \bar{u}(k_1) \otimes \bar{u}(k'_1)) \mathbb{P} \left( I_0(\vec{p}, \vec{p}') \mathbb{P} \right)^{n-1} u(p_1) \otimes u(p'_1) \delta_+(p_1, p'_1, k_1, k'_1). \tag{B.15}$$

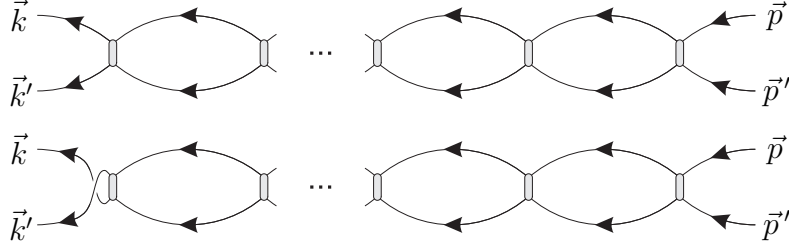


Figure 11: **Higher loop diagrams in AAF model.**

Now we take into account the derivatives. The corresponding situation in the LL model was particularly lucky as any distribution of the derivatives led to precisely the same expression. Here, this is not the case and we are confronted with an rather large combinatorial problem. As all cases have to be handled separately anyway, it turns out to be more convenient to even consider all orders in  $g$  at once. The entire higher loop series is written in four terms:

- no derivatives on external legs

$$\begin{aligned} & (\bar{u}(k_1) \otimes \bar{u}(k'_1) - \bar{u}(k'_1) \otimes \bar{u}(k_1)) V_{10} u(p_1) \otimes u(p'_1) \delta_+(p_1, p'_1, k_1, k'_1) \\ & = (\bar{u}(p_1) \otimes \bar{u}(p'_1) - \bar{u}(p'_1) \otimes \bar{u}(p_1)) V_{10} u(p_1) \otimes u(p'_1) \delta_-(p_1, p'_1, k_1, k'_1) \end{aligned} \quad (\text{B.16a})$$

- derivatives only on the in-going legs

$$\begin{aligned} & (\vec{p}' \times \vec{p}) (\bar{u}(k_1) \otimes \bar{u}(k'_1) - \bar{u}(k'_1) \otimes \bar{u}(k_1)) V_{11} u(p_1) \otimes u(p'_1) \delta_+(p_1, p'_1, k_1, k'_1) \\ & = (\vec{p}' \times \vec{p}) (\bar{u}(p_1) \otimes \bar{u}(p'_1) - \bar{u}(p'_1) \otimes \bar{u}(p_1)) V_{11} u(p_1) \otimes u(p'_1) \delta_-(p_1, p'_1, k_1, k'_1) \end{aligned} \quad (\text{B.16b})$$

- derivatives only on the out-going legs

$$\begin{aligned} & (\vec{k}' \times \vec{k}) (\bar{u}(k_1) \otimes \bar{u}(k'_1) + \bar{u}(k'_1) \otimes \bar{u}(k_1)) V_{00} u(p_1) \otimes u(p'_1) \delta_+(p_1, p'_1, k_1, k'_1) \\ & = (\vec{p}' \times \vec{p}) (\bar{u}(p_1) \otimes \bar{u}(p'_1) + \bar{u}(p'_1) \otimes \bar{u}(p_1)) V_{00} u(p_1) \otimes u(p'_1) \delta_-(p_1, p'_1, k_1, k'_1) \end{aligned} \quad (\text{B.16c})$$

- derivatives on the in- and out-going legs

$$\begin{aligned} & (\vec{p}' \times \vec{p}) (\vec{k}' \times \vec{k}) (\bar{u}(k_1) \otimes \bar{u}(k'_1) + \bar{u}(k'_1) \otimes \bar{u}(k_1)) V_{01} u(p_1) \otimes u(p'_1) \delta_+(p_1, p'_1, k_1, k'_1) \\ & = (\vec{p}' \times \vec{p})^2 (\bar{u}(p_1) \otimes \bar{u}(p'_1) + \bar{u}(p'_1) \otimes \bar{u}(p_1)) V_{01} u(p_1) \otimes u(p'_1) \delta_-(p_1, p'_1, k_1, k'_1) \end{aligned} \quad (\text{B.16d})$$

It remains to find expressions for the  $V$ 's which sum all possible diagrams. The possible placements of the derivatives along the chain of bubbles are in one-to-one correspondence with binary sequences. One sequence of length  $n$  corresponds to one Feynman diagram at order  $g^n$ . Let “0” stand for derivatives “on the left” of the vertex (i.e. towards



out-going particles) and “1” for derivatives “on the right” of the vertex (i.e. towards in-going particles). For a diagram without derivatives on the external legs this sequence has to begin with a “1” and end with a “0”, hence the notation  $V_{10}$ , etc. Now, this binary sequence translates into a sequence of bubble propagators  $I_0$  (no derivatives),  $I_1$  (one pair of derivatives) and  $I_2$  (two pairs of derivatives) according to the rules

$$\begin{aligned} \dots 00 \dots &\mapsto \dots (-I_1) \dots \\ \dots 01 \dots &\mapsto \dots (+I_0) \dots \\ \dots 10 \dots &\mapsto \dots (-I_2) \dots \\ \dots 11 \dots &\mapsto \dots (+I_1) \dots \end{aligned} \tag{B.17}$$

where we have chosen to include the sign from the second vertex (Recall that the vertex with the derivatives on the left comes with a minus sign, (4.11)). The sign for the very first digit in the binary sequence has to be taken into account separately.

The next step is to generate all those sequences. The main idea is to think of an arbitrary binary sequence as a succession of constant subsequences, e.g. the sequence 1111000100011111000 is made of 6 constant subsequences. It is clear, that all sequences are obtained from joining an arbitrary number of constant subsequences of arbitrary length. The constant pieces lead to

$$\mathbb{P}I_{\pm} := \sum_{n=0}^{\infty} G^n (\pm \mathbb{P}I_1)^n, \tag{B.18}$$

where the minus sign is used for a sequence of zeros and the plus sign for a sequence of ones. In (B.18) we included for each vertex a matrix  $\mathbb{P}$  and one power of the coupling constant  $G = \frac{-ig}{m^2}$ . Between these constant subsequences we have to insert a factor  $G\mathbb{P}I_0$  when the sequence changes from “0” to “1” and a factor  $-G\mathbb{P}I_2$  when the sequence changes from “1” to “0”. Taking all this together we find

$$V_{10} := + \sum_{k=0}^{\infty} G^{2k+2} \mathbb{P}I_+ (-\mathbb{P}I_2) \mathbb{P}I_- (-\mathbb{P}I_0 \mathbb{P}I_+ \mathbb{P}I_2 \mathbb{P}I_-)^k \mathbb{P}, \tag{B.19a}$$

$$V_{11} := + \sum_{k=0}^{\infty} G^{2k+1} \mathbb{P}I_+ (-\mathbb{P}I_2 \mathbb{P}I_- \mathbb{P}I_0 \mathbb{P}I_+)^k \mathbb{P}, \tag{B.19b}$$

$$V_{00} := - \sum_{k=0}^{\infty} G^{2k+1} \mathbb{P}I_- (-\mathbb{P}I_0 \mathbb{P}I_+ \mathbb{P}I_2 \mathbb{P}I_-)^k \mathbb{P}, \tag{B.19c}$$

$$V_{01} := - \sum_{k=0}^{\infty} G^{2k+2} \mathbb{P}I_- (+\mathbb{P}I_0) \mathbb{P}I_+ (-\mathbb{P}I_2 \mathbb{P}I_- \mathbb{P}I_0 \mathbb{P}I_+)^k \mathbb{P}. \tag{B.19d}$$

The sums in (B.18) and (B.19) can be evaluated explicitly as functions of  $\vec{p}$ ,  $\vec{p}'$  and  $m$ . It is computationally convenient to replace everywhere

$$I_k \rightarrow \tilde{I}_k = \left(\frac{1}{2}\mathbb{P}\right)^2 I_k \left(\frac{1}{2}\mathbb{P}\right)^2. \tag{B.20}$$

This is possible because every  $I_k$  is sandwiched between two  $\mathbb{P}$ 's which satisfy the property (B.14). We refrain from printing the explicit  $V$ 's as the formulas are rather lengthy and

not very illuminating. However, after inserting them into (B.16), evaluating the spinor products and multiplying everything by the Jacobian  $p'_0 p_0 / \vec{p}' \times \vec{p}$  (cf. (4.17)) we find the fairly compact result

$$\sum_{n=1}^{\infty} \langle k k' | \hat{S} | p p' \rangle \Big|_{g^n} = \frac{2 \frac{ig}{m^2} \vec{p}' \times \vec{p}}{1 - \frac{ig}{m^2} \vec{p}' \times \vec{p}} \delta_{-}(p_1, p'_1, k_1, k'_1) . \quad (\text{B.21})$$

## C Computational details in FR model

### C.1 Loop integrals

We need to compute

$$\text{Loop} = - \int \frac{d^2 k}{(2\pi)^2} \frac{\not{p} + \not{p}' - \not{k} + m}{(p + p' - k)^2 - m^2} \otimes \frac{\not{k} + m}{k^2 - m^2} \quad (\text{C.1})$$

with the non-relativistic pole prescription. Performing the integral over  $k^0$  first we get

$$\text{Loop} = \int_{-\infty}^{+\infty} \frac{dk}{2\pi i} \frac{P(k; p, p')}{2(p + p')^2 (k - p_1 - i\epsilon)(k - p'_1 + i\epsilon)} . \quad (\text{C.2})$$

The denominator  $P(k; p, p')$  is a cubic polynomial in  $k$  which is symmetric in  $p$  and  $p'$ :

$$\begin{aligned} P(k; p, p') &= 2E [E\gamma^0 - (P - k)\gamma^1 + m] \otimes (m - k\gamma^1) \\ &\quad - [E^2 + k^2 - (P - k)^2] \gamma^0 \otimes (m - k\gamma^1) \\ &\quad + [E^2 + k^2 - (P - k)^2] [E\gamma^0 - (P - k)\gamma^1 + m] \otimes \gamma^0 \\ &\quad - 2E(k^2 + m^2) \gamma^0 \otimes \gamma^0 , \end{aligned} \quad (\text{C.3})$$

where  $E = p_0 + p'_0$ ,  $P = p_1 + p'_1$ . Its most important property is that

$$\begin{aligned} P(p_1; p, p') &= (\not{p}' + m) \otimes (\not{p} + m) , \\ P(p'_1; p, p') &= (\not{p} + m) \otimes (\not{p}' + m) . \end{aligned} \quad (\text{C.4})$$

The integral in (C.2) diverges and requires regularization. Any reasonable regularization (for instance dimensional regularization) closes the contour of integration symmetrically in the lower and upper half-planes. Evaluating the residues at  $k = p_1$ ,  $k = p'_1$  we get eq. (5.20) in the main text.

## References

- [1] J. M. Maldacena: *The large  $N$  limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231, [hep-th/9711200](#) • S. S. Gubser, I. R. Klebanov and A. M. Polyakov: *Gauge theory correlators from non-critical string theory*, Phys. Lett. **B428** (1998) 105, [hep-th/9802109](#) • E. Witten: *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253, [hep-th/9802150](#).

- [2] R. R. Metsaev and A. A. Tseytlin: *Type IIB superstring action in  $\text{AdS}_5 \times S^5$  background*, Nucl. Phys. B **533**, 109 (1998), [hep-th/9805028](#).
- [3] I. Bena, J. Polchinski and R. Roiban: *Hidden symmetries of the  $\text{AdS}_5 \times S^5$  superstring*, Phys. Rev. D **69** (2004) 046002, [hep-th/0305116](#).
- [4] N. Beisert: *The dilatation operator of  $\mathcal{N} = 4$  super Yang-Mills theory and integrability*, Phys. Rept. **405**, 1 (2005), [hep-th/0407277](#) • N. Beisert: *Higher-loop integrability in  $\mathcal{N} = 4$  gauge theory*, Comptes Rendus Physique **5**, 1039 (2004), [hep-th/0409147](#) • J. Plefka: *Spinning strings and integrable spin chains in the  $\text{AdS}/\text{CFT}$  correspondence*, [hep-th/0507136](#).
- [5] K. Zarembo: *Semiclassical Bethe ansatz and  $\text{AdS}/\text{CFT}$* , Comptes Rendus Physique **5**, 1081 (2004) [Fortsch. Phys. **53**, 647 (2005)], [hep-th/0411191](#).
- [6] J. A. Minahan and K. Zarembo: *The Bethe-ansatz for  $\mathcal{N} = 4$  super Yang-Mills*, JHEP **0303**, 013 (2003), [hep-th/0212208](#).
- [7] N. Beisert, C. Kristjansen and M. Staudacher: *The dilatation operator of  $\mathcal{N} = 4$  super Yang-Mills theory*, Nucl. Phys. B **664**, 131 (2003), [hep-th/0303060](#) • N. Beisert and M. Staudacher: *The  $\mathcal{N} = 4$  SYM integrable super spin chain*, Nucl. Phys. B **670**, 439 (2003) [hep-th/0307042](#) • N. Beisert, V. Dippel and M. Staudacher: *A novel long range spin chain and planar  $\mathcal{N} = 4$  super Yang-Mills*, JHEP **0407**, 075 (2004), [hep-th/0405001](#).
- [8] N. Beisert and M. Staudacher: *Long-range  $\text{PSU}(2,2|4)$  Bethe ansatz for gauge theory and strings*, Nucl. Phys. B **727**, 1 (2005), [hep-th/0504190](#).
- [9] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo: *Classical / quantum integrability in  $\text{AdS}/\text{CFT}$* , JHEP **0405**, 024 (2004), [hep-th/0402207](#).
- [10] V. A. Kazakov and K. Zarembo: *Classical / quantum integrability in non-compact sector of  $\text{AdS}/\text{CFT}$* , JHEP **0410**, 060 (2004), [hep-th/0410105](#) • N. Beisert, V. A. Kazakov and K. Sakai: *Algebraic curve for the  $\text{SO}(6)$  sector of  $\text{AdS}/\text{CFT}$* , [hep-th/0410253](#) • S. Schäfer-Nameki: *The algebraic curve of 1-loop planar  $\mathcal{N} = 4$  SYM*, Nucl. Phys. B **714**, 3 (2005), [hep-th/0412254](#) • N. Beisert, V. A. Kazakov, K. Sakai and K. Zarembo: *The algebraic curve of classical superstrings on  $\text{AdS}_5 \times S^5$* , [hep-th/0502226](#).
- [11] N. Dorey and B. Vicedo: *On the dynamics of finite-gap solutions in classical string theory*, [hep-th/0601194](#).
- [12] S. S. Gubser, I. R. Klebanov and A. M. Polyakov: *A semi-classical limit of the gauge/string correspondence* Nucl. Phys. **B636** (2002) 99, [hep-th/0204051](#) • S. Frolov and A. A. Tseytlin: *Multi-spin string solutions in  $\text{AdS}_5 \times S^5$* , Nucl. Phys. **B668** (2003) 77, [hep-th/0304255](#) • S. Frolov and A. A. Tseytlin: *Rotating string solutions:  $\text{AdS}/\text{CFT}$  duality in non-supersymmetric sectors*, Phys. Lett. B **570**, 96 (2003), [hep-th/0306143](#) • A. A. Tseytlin: *Semiclassical strings and  $\text{AdS}/\text{CFT}$* , [hep-th/0409296](#).

- [13] H. Bethe: *On The Theory Of Metals. 1. Eigenvalues And Eigenfunctions For The Linear Atomic Chain*, Z. Phys. **71**, 205 (1931).
- [14] L. D. Faddeev: *How Algebraic Bethe Ansatz works for integrable model*, hep-th/9605187.
- [15] V.E. Korepin, A.G. Izergin and N.M. Bogolyubov: *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz* (Cambridge Univ. Press, 1992).
- [16] A. B. Zamolodchikov and A. B. Zamolodchikov: *Factorized S-Matrices In Two Dimensions As The Exact Solutions Of Certain Relativistic Quantum Field Models*, Annals Phys. **120** (1979) 253.
- [17] A. Rej, D. Serban and M. Staudacher: *Planar  $\mathcal{N} = 4$  gauge theory and the Hubbard model*, hep-th/0512077.
- [18] C. G. Callan, H. K. Lee, T. McLoughlin, J. H. Schwarz, I. Swanson and X. Wu: *Quantizing string theory in  $\text{AdS}_5 \times S^5$ : Beyond the pp-wave*, Nucl. Phys. B **673**, 3 (2003), hep-th/0307032 • C. G. Callan, T. McLoughlin and I. J. Swanson: *Holography beyond the Penrose limit*, Nucl. Phys. B **694**, 115 (2004), hep-th/0404007 • C. G. Callan, T. McLoughlin and I. J. Swanson: *Higher impurity AdS/CFT correspondence in the near-BMN limit*, Nucl. Phys. B **700**, 271 (2004), hep-th/0405153 • T. McLoughlin and I. J. Swanson: *N-impurity superstring spectra near the pp-wave limit*, Nucl. Phys. B **702**, 86 (2004), hep-th/0407240.
- [19] G. Arutyunov, S. Frolov and M. Staudacher: *Bethe ansatz for quantum strings*, JHEP **0410**, 016 (2004), hep-th/0406256.
- [20] M. Staudacher: *The factorized S-matrix of CFT/AdS*, JHEP **0505**, 054 (2005), hep-th/0412188.
- [21] S. Frolov, J. Plefka and M. Zamaklar: *The  $\text{AdS}_5 \times S^5$  superstring in light-cone gauge and its Bethe equations*, hep-th/0603008.
- [22] N. Beisert and A. A. Tseytlin: *On quantum corrections to spinning strings and Bethe equations*, Phys. Lett. B **629**, 102 (2005), hep-th/0509084.
- [23] N. Mann and J. Polchinski: *Bethe ansatz for a quantum supercoset sigma model*, Phys. Rev. D **72**, 086002 (2005), hep-th/0508232.
- [24] A. M. Polyakov: *Supermagnets and sigma models*, hep-th/0512310.
- [25] L. Freyhult, C. Kristjansen and T. Månsson: *Integrable spin chains with  $U(1)^3$  symmetry and generalized Lunin-Maldacena backgrounds*, JHEP **0512**, 008 (2005), hep-th/0510221.
- [26] N. Beisert: *The  $su(2|2)$  dynamic S-matrix*, hep-th/0511082.

- [27] L. D. Faddeev, E. K. Sklyanin and L. A. Takhtajan: *The Quantum Inverse Problem Method. 1*, Theor. Math. Phys. **40**, 688 (1980) [Teor. Mat. Fiz. **40**, 194 (1979)].
- [28] M. Kruczenski: *Spin chains and string theory*, hep-th/0311203 • M. Kruczenski, A. V. Ryzhov and A. A. Tseytlin: *Large spin limit of  $\text{AdS}_5 \times S^5$  string theory and low energy expansion of ferromagnetic spin chains*, Nucl. Phys. B **692**, 3 (2004), hep-th/0403120.
- [29] L. F. Alday, G. Arutyunov and S. Frolov: *New integrable system of 2dim fermions from strings on  $\text{AdS}_5 \times S^5$* , JHEP **0601**, 078 (2006), hep-th/0508140.
- [30] L. D. Faddeev and N. Y. Reshetikhin: *Integrability Of The Principal Chiral Field Model In (1+1)-Dimension*, Annals Phys. **167** (1986) 227.
- [31] A. M. Polyakov and P. B. Wiegmann: *Theory Of Nonabelian Goldstone Bosons In Two Dimensions*, Phys. Lett. B **131**, 121 (1983).
- [32] H. B. Thacker: *Exact Integrability In Quantum Field Theory And Statistical Systems*, Rev. Mod. Phys. **53**, 253 (1981).
- [33] M. Lakshmanan: *Continuum spin system as an exactly solvable dynamical system*, Phys. Lett. A **61**, 53 (1977) • L.A. Takhtajan: *Integration Of The Continuous Heisenberg Spin Chain Through The Inverse Scattering Method*, Phys. Lett. A **64**, 235 (1977) • V. E. Zakharov and L. A. Takhtajan: *Equivalence Of The Nonlinear Schrödinger Equation And The Equation Of A Heisenberg Ferromagnet*, Theor. Math. Phys. **38**, 17 (1979) [Teor. Mat. Fiz. **38**, 26 (1979)] • R.F. Bikbaev, A.I. Bobenko and A.R. Its: *Finite-zone integration of the Landau-Lifshitz equation*, Dokl. Akad. Nauk SSSR **272** (1983) 1293.
- [34] E.K. Sklyanin: *Quantization of the Continuous Heisenberg Ferromagnet*, Lett. Math. Phys. **15** (1988) 357.
- [35] H. B. Thacker: *Bethe's Hypothesis And Feynman Diagrams: Exact Calculation Of A Three Body Scattering Amplitude By Perturbation Theory*, Phys. Rev. D **11**, 838 (1975) • H. B. Thacker: *Many Body Scattering Processes In A One-Dimensional Boson System*, Phys. Rev. D **14**, 3508 (1976).
- [36] J. A. Minahan, A. Tirziu and A. A. Tseytlin:  *$1/J$  corrections to semiclassical  $\text{AdS}/\text{CFT}$  states from quantum Landau-Lifshitz model*, Nucl. Phys. B **735**, 127 (2006), hep-th/0509071 • A. Tirziu: *Quantum Landau-Lifshitz model at four loops:  $1/J$  and  $1/J^2$  corrections to BMN energies*, hep-th/0601139.
- [37] J. A. Minahan, A. Tirziu and A. A. Tseytlin:  *$1/J^2$  corrections to BMN energies from the quantum long range Landau-Lifshitz model*, JHEP **0511**, 031 (2005), hep-th/0510080.
- [38] L.D. Faddeev and L.A. Takhtajan: *Hamiltonian methods in the theory of solitons* (Springer-Verlag, 1987).

- [39] B. Sutherland: *Low-Lying Eigenstates of the One-Dimensional Heisenberg Ferromagnet for any Magnetization and Momentum*, Phys. Rev. Lett. **74**, 816 (1995) • A. Dhar and B.S. Shastry: *Bloch Walls and Macroscopic String States in Bethe's solution of the Heisenberg Ferromagnetic Linear Chain*, cond-mat/0005397 • N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo: *Stringing spins and spinning strings*, JHEP **0309** (2003) 010, hep-th/0306139.
- [40] G. Arutyunov and S. Frolov: *Uniform light-cone gauge for strings in  $\text{AdS}_5 \times S^5$ : Solving  $su(1|1)$  sector*, JHEP **0601**, 055 (2006), hep-th/0510208.
- [41] F.A. Berezin and V.N. Sushko: *Relativistic two-dimensional model of a self-interacting fermion field with nonzero mass in the state of rest*, Sov. Phys. JETP **21** (1965) 865 [Zh. Eksp. Teor. Fiz. **48** (1965) 1293].
- [42] H. Bergknoff and H. B. Thacker: *Method For Solving The Massive Thirring Model*, Phys. Rev. Lett. **42**, 135 (1979) • H. Bergknoff and H. B. Thacker: *Structure And Solution Of The Massive Thirring Model*, Phys. Rev. D **19**, 3666 (1979).
- [43] V. E. Korepin: *Direct Calculation Of The S Matrix In The Massive Thirring Model*, Theor. Math. Phys. **41**, 953 (1979) [Teor. Mat. Fiz. **41**, 169 (1979)].
- [44] V. E. Korepin: *New Effects In The Massive Thirring Model: Repulsive Case*, Commun. Math. Phys. **76**, 165 (1980).
- [45] A. Mikhailov: *A nonlocal Poisson bracket of the sine-Gordon model*, hep-th/0511069.
- [46] D. Berenstein, J. M. Maldacena and H. Nastase: *Strings in flat space and pp waves from  $\mathcal{N} = 4$  Super Yang Mills*, JHEP **0204** (2002) 013, hep-th/0202021.
- [47] N. Gromov, V. Kazakov, K. Sakai and P. Vieira: *Strings as Multi-particle States of Quantum Sigma-Models*, to appear.